Computing quadratic points on $X_0(N)$

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Rational points on modular curves

Torsion primes: $X_1(p)(\mathbf{Q})$

Theorem (Mazur 1977)

Let $E/\mathbf{Q}$ be an elliptic curve. If $x \in E(\mathbf{Q})_{\mathrm{tors}}$, then the support of $\operatorname{ord}(x)$ is contained in $\{2,3,5,7\}$.

The proof computes the non-cuspidal points in $X_1(p)(\mathbf{Q})$ for all p.

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Let $E/\mathbf{Q}$ be an elliptic curve. If E has a cyclic p-isogeny, then $p \in \{2, 3, 5, 7, 11, 13, 17, 19, 37, 43, 67, 163\}$.

If E is in addition non-CM, then $p \leq 37$.

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Higher degree fields: $X_1(p)(K)$

Theorem (Merel, Kamienny, Oesterlé)

Let K be a number field of degree d.

Then $Y_1(p)(K) = \emptyset$ if $p > (3^{d/2}+1)^2$.

Complete computation of degree d points for:

- $d = 2$: Kamienny
- $d = 3$: Derickx–Etropolski–van Hoeij–Morrow–Zureick-Brown
- $4 \leq d \leq 7$: Derickx–Kamienny–Stein–Stoll
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Higher degree fields: $X_0(N)(K)$

Harder: CM point in $X_0(p)(K)$ for p split in K with $h_K = 1$.

Conjecture

If $N > C(d)$, then $X_0(N)(K)$ consists only of cusps and CM points for all K of degree d.

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- all $N = p$, $h_K \neq 1$, conditional on GRH (Banwait–Derickx 2022)
- all $N = p$ for $K = \mathbf{Q}(\sqrt{d})$ with $d = -5, 2, 3, 5, 6, 7$ and for semistable E (Michaud-Jacobs 2022)
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Quadratic points on $X_0(N)$ of small genus

Known results for fixed N

Problem for describing quadratic points on X of genus ≥ 2: there can be infinitely many, namely iff X is hyperelliptic or $X \to E$ with $\operatorname{rk} E(\mathbf{Q}) > 0$ (Harris–Silverman).

For the following N, the quadratic points on $X_0(N)$ had previously been computed:

- $X_0(N)$ hyperelliptic, rank 0: Bruin–Najman
- $X_0(N)$ non-hyperelliptic of genus ≤ 5, rank 0: Ozman–Siksek
- $X_0(N)$ of genus ≤ 5, rank > 0: Box
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- some other $X_0(N)$: $N = 77, 91, 125, 169$.

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At an MIT workshop on modular curves (aim: extend the LMFDB) we started to extend these computations to push to highest possible genus of $X_0(N)$ by improving state-of-the-art methods:

Theorem

For all $X_0(N)$ of genus ≤ 8 (N composite) and ≤ 10 (N prime), the (finitely many) quadratic points on $X_0(N)$ are only cusps and CM points, except for $N = 103$ ($g = 8$) and a point over $\mathbf{Q}(\sqrt{2885})$.

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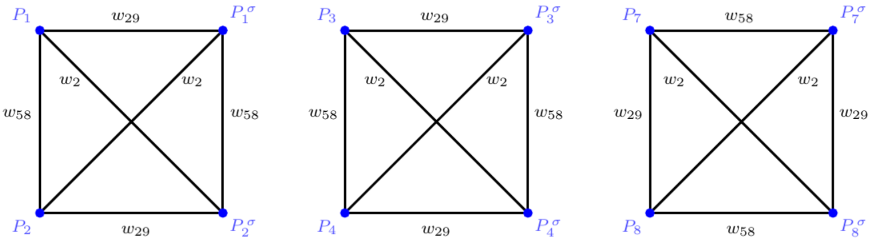
Example: $X_0(58)$

Point	Field	j-invariant	CM
P_1	$\mathbb{Q}(\sqrt{-1})$	1728	-4
P_2	$\mathbb{Q}(\sqrt{-1})$	287496	-16
P_3	$\mathbb{Q}(\sqrt{-7})$	-3375	-7
P_4	$\mathbb{Q}(\sqrt{-7})$	16581375	-28
P_5	$\mathbb{Q}(\sqrt{-1})$	1728	-4
P_6	$\mathbb{Q}(\sqrt{29})$	$-561477670097798464000\sqrt{29} + 3023649789249456720000$	-232
P_7	$\mathbb{Q}(\sqrt{-7})$	-3375	-7
P_8	$\mathbb{Q}(\sqrt{-7})$	-3375	-7



Our methods

Computing models of $X_0(N)/W'(N)$ and the j-map

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- $j\colon X_0(N) \to \mathbf{P}^1$ using q-expansions up to $O(q^m)$ with the (easy to compute) bound

$$m = (2g-2)r + 1 + \deg(j)$$

and

$$r > \frac{\deg(j)}{2(g-1)} + \frac{1}{2}, \quad \deg(j) = N \prod_{p \mid N} \Big(1 + \frac{1}{p}\Big).$$

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Use finite morphisms $X_0(N) \to X$
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- $X_0(2 \cdot 29)/w_{29}$ with $r < g$ (Chabauty) or
- $X_0(2 \cdot 2 \cdot 29)^+$: only cusps and CM points.

Result: $2^{\omega(N)}$ cusps, 7 CM points with $j \in \mathbf{Q}$,
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Rank 0

Assume $J_0(N)(\mathbf{Q})$ is finite and we know $I \in \mathbf{Z}_{\geq 1}$ with $I \cdot J_0(N)(\mathbf{Q}) \subseteq C_0(N)(\mathbf{Q})$, the cuspidal divisor class group, e.g. from bound on $J_0(N)(\mathbf{Q})_{\mathrm{tors}}$ from reductions modulo p's.

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using the Derickx formal immersion criterion.

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Assume there is d with $J(\mathbf{Q}) = (1 + w_d)J(\mathbf{Q}) \oplus (1 - w_d)J(\mathbf{Q})$ (up to 2-torsion) with $(1 - w_d)J(\mathbf{Q}) \subseteq J(\mathbf{Q})_{\mathrm{tors}}$.

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