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# Exact verification of the strong BSD conjecture for some absolutely simple modular abelian surfaces

see arXiv:2107.00325 and forthcoming articles

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joint with Michael Stoll

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(previously Universität Bayreuth)

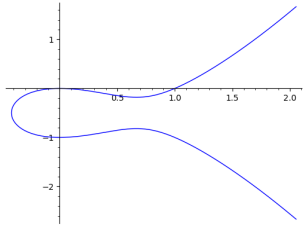
July 5, 2022

# The BSD conjecture: Why is it useful?

# Introduction:
# How to compute the $\mathbf{Q}$-points of a given genus 1 curve?

- Consider

$$E : y^2 + y = x^3 - x^2 \quad \text{(model of } X_1(11)\text{, 11a3)}$$

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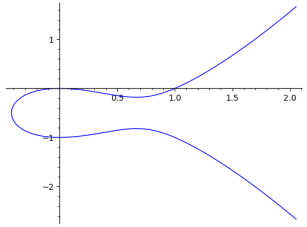

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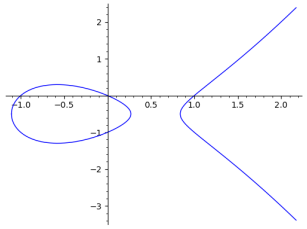

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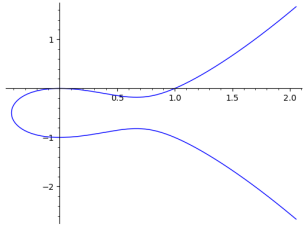

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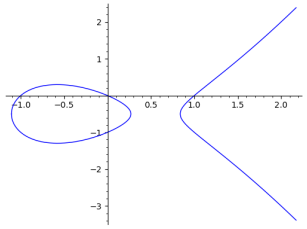

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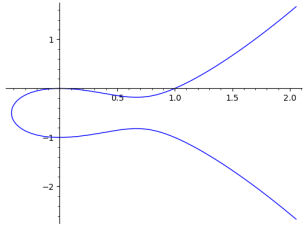

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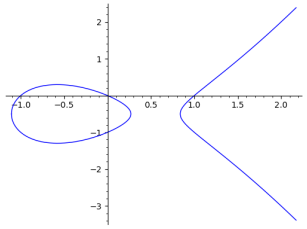

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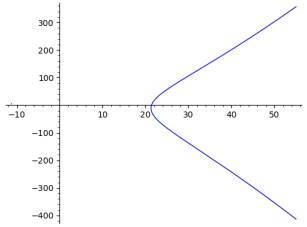

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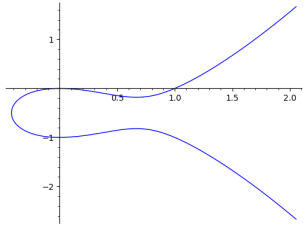

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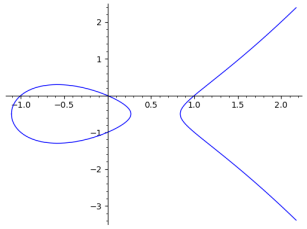

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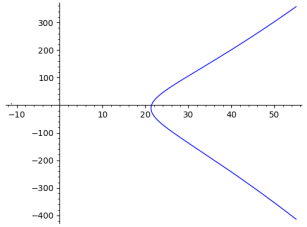

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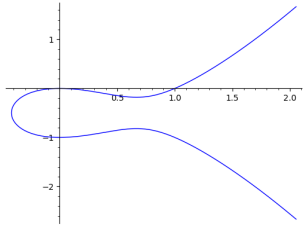

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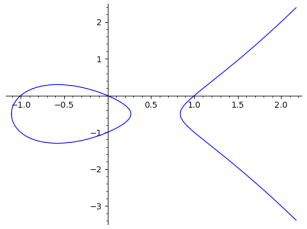

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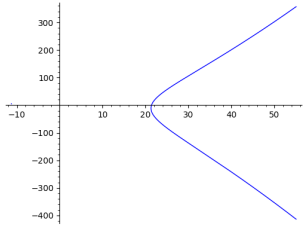

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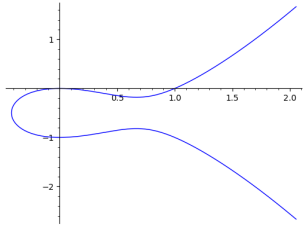

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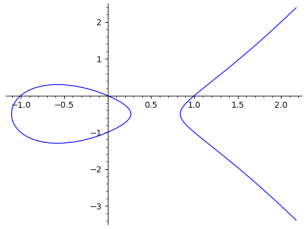

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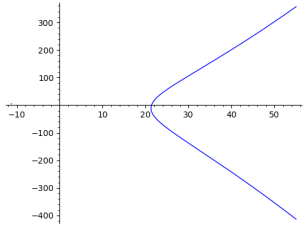

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# How to compute the rank?

**Theorem of Mordell–Weil**

Let $A$ be an abelian variety over $\mathbf{Q}$, e.g. $A = E$ an elliptic curve.
Then the Mordell–Weil group $A(\mathbf{Q})$ is a finitely generated abelian group.

**Proof (sketch).**

For every $n > 1$, there is an $n$-descent exact sequence

$$0 \to A(\mathbf{Q})/n \to \mathrm{Sel}_n(A/\mathbf{Q}) \to Ш(A/\mathbf{Q})[n] \to 0 \tag{1}$$

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with $a_p := p + 1 - \#E(\mathbf{F}_p)$ for $p \in S_{\text{good}}$ (trace of $p$-Frobenius).

**Birch–Swinnerton-Dyer (rank) conjecture**

$$r = r_{\text{an}} := \mathrm{ord}_{s=1} L(E,s)$$

- Formulated based on computations in 1965.
- $r_{\text{an}}$ well-defined by modularity of $E/\mathbf{Q}$.
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classifies everywhere locally trivial $A$-torsors.

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# Can you give a non-trivial example for Ш?

The Selmer cubic

$$C : 3x^3 + 4y^3 + 5z^3 = 0$$

is a non-trivial element of $Ш(E/\mathbf{Q})[3]$ for the elliptic curve

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**Proof (sketch).**

- $C(\mathbf{Q}_v) \neq \emptyset$: Weil conjectures or $\mathrm{H}^1(\mathbf{F}_p, E) = 0$ and Hensel's lemma.
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# What are applications of the strong BSD conjecture?

**Problem**

Let $C/\mathbf{Q}$ be a curve of genus 1. Decide: $C(\mathbf{Q}) = \emptyset$? $\#C(\mathbf{Q}) = \infty$?

- Compute elliptic curve $E/\mathbf{Q}$ such that $[C] \in Ш(E/\mathbf{Q})$.
- If one can decide $C(\mathbf{Q}) \neq \emptyset$, one can decide $\#C(\mathbf{Q}) = \infty$ by deciding $L(E, 1) = 0$ (BSD rank conjecture).
- Compute $\#Ш(E/\mathbf{Q})$ using strong BSD.
- Enumerate representatives of $Ш(E/\mathbf{Q})$.
- Use the perfect Cassels–Tate pairing

$$\langle \cdot, \cdot \rangle : Ш(E/\mathbf{Q}) \times Ш(E/\mathbf{Q}) \to \mathbf{Q}/\mathbf{Z}$$

to decide existence of $[D] \in Ш(E/\mathbf{Q})$ with $\langle [C], [D] \rangle \neq 0$.

Questions?

# The BSD conjecture: What is known?

# What is an isogeny?
# How is it related to Galois representations?

**Definition**

An isogeny of abelian varieties is a finite surjective morphism.

- For $p$ prime, one has an isogeny $[p] : A \to A$.
- $A[p]$ is a finite flat $p$-torsion group scheme of order $p^{2\dim A}$.
- Sometimes $[p]$ can be factorized. This holds if and only if there is a non-trivial subgroup scheme of $A[p]$. In this case,

$$\rho_p : \mathrm{Gal}(\overline{\mathbf{Q}}|\mathbf{Q}) \to \mathrm{Aut}(A[p])$$

  is reducible and vice versa.
- If $A = E$ is an elliptic curve, reducible $\rho_p$ look like:

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- The functor representing (generalized) elliptic curves with a cyclic $N$-isogeny (up to twist) is representable by a nice curve, called $X_0(N)$.
- Its Jacobian is denoted by $J_0(N)$.
- The space $S_2(\Gamma_0(N))$ of cusp forms is $\mathrm{H}^0(X_0(N), \Omega^1)$.

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- The space $S_2(\Gamma_0(N))$ of cusp forms is $\mathrm{H}^0(X_0(N), \Omega^1)$.

- $\dim_{\mathbf{C}} S_2(\Gamma_0(N)) = g(X_0(N)) < \infty$
- There is an injective $q$-expansion morphism $S_2(\Gamma_0(N)) \hookrightarrow \mathbf{C}[\![q]\!], f \mapsto \sum_{n\geq 1} a_n(f) q^n$.
- Example: $S_2(\Gamma_0(11)) = \mathbf{C} \cdot (\Delta(z)\Delta(11z))^{\frac{1}{12}}$ with

$$\Delta(z) := (2\pi)^{12} \cdot q \prod_{n\geq 1} (1 - q^n)^{24}, \quad q = e^{2\pi i z}.$$

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## Definition

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# What is a modular abelian variety?

**Theorem (consequence of Serre's Modularity Conjecture)**

Let $A/\mathbf{Q}$ be an absolutely simple abelian variety.
The following are equivalent:

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$\mathbf{Q}(f) \simeq \mathrm{End}^0_{\mathbf{Q}}(A)$,
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# What is important to remember for this talk?

Let $A$ be a modular abelian variety[1] over $\mathbf{Q}$ with associated newform $f$.

- Assume that $\mathrm{ord}_{s=1} L(f,s) \in \{0,1\}$ (hence $r_{\mathrm{an}} \in \{0, \dim A\}$).
- This implies by combining the Gross–Zagier formula with the Heegner point Euler system of Kolyvagin–Logachëv:

$$r = r_{\mathrm{an}}, \quad \text{(BSD rank conjecture)}$$
$$\#Ш(A/\mathbf{Q}) < \infty,$$
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# In which cases has strong BSD been verified?

- For elliptic curves with $r_{\mathrm{an}} \leq 1$:
  - Strong BSD verified exactly for levels $N < 5000$ combining work of Grigorov–Jorza–Patrikis–Stein–Tarniţă (2009), Miller (2011), Miller–Stoll (2013, isogeny descent), Creutz–Miller (2012, second isogeny descent), Lawson–Wuthrich (2016, use of $p$-adic $L$-functions).
- For modular abelian varieties of dimension $> 1$:
  - Flynn–Leprévost–Schaefer–Stein–Stoll–Wetherell (2001): BSD for some Jacobians of dimension 2 numerically.
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Questions?

# What are our new results in dimension 2?

# How to bound $\#Ш(A/\mathbf{Q})$?

There are two reasons why $Ш(A)$ could be infinite:

- "horizonal": $Ш(A)[p] \neq 0$ for infinitely many $p$.
- "vertical": $Ш(A)[p^\infty] \cong F \oplus (\mathbf{Q}_p/\mathbf{Z}_p)^n$ infinite for one $p$.

Solution to the "horizontal" problem:

**Theorem (K.): explicit Euler system of Kolyvagin–Logachëv**

Let $A$ be a modular abelian variety over $\mathbf{Q}$. Denote $\mathcal{O} := \mathrm{End}_{\mathbf{Q}}(A)$.
One has $Ш(A/\mathbf{Q})[\mathfrak{p}] = 0$ for all $\mathfrak{p}$ with

- $\rho_{\mathfrak{p}} : \mathrm{Gal}(\overline{\mathbf{Q}}|\mathbf{Q}) \to \mathrm{Aut}_{\mathbf{F}_{\mathfrak{p}}}(A[\mathfrak{p}](\overline{\mathbf{Q}}))$ irreducible and
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# What are the main obstacles in dimension > 1?

**Problems when dim $A > 1$ (necessary input for Euler system)**

- We don't have an analog of Mazur's classification of rational isogenies of prime degree for *all* $A$: moduli spaces have dimension > 1.
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# How to compute the remaining $Ш(A/\mathbf{Q})[\mathfrak{p}^\infty]$?

Two tools:

- Perform a $\mathfrak{p}^n$-descent to compute $\mathrm{Sel}_{\mathfrak{p}^n}(A/\mathbf{Q})$.
  - Works very well if $\rho_{\mathfrak{p}^n}$ is reducible.
  - Works for general $\mathfrak{p}^n$ in principle, but:
  - Infeasible if $\rho_{\mathfrak{p}^n}$ has large image, e.g., $\#\mathcal{O}/\mathfrak{p}^n > 7$ and $\rho_{\mathfrak{p}^n}$ irreducible, even assuming GRH.
- Compute the $\mathfrak{p}$-adic $L$-function and use the $\mathrm{GL}_2$ IMC.
  - Can be computed very efficiently with overconvergent modular symbols using the Pollack–Stevens–Greenberg algorithm.
  - Requires $\rho_{\mathfrak{p}}$ to be irreducible. (But: work in progress joint with Castella)
  - Unclear for good non-ordinary and especially bad non-multiplicative reduction.
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# How do we verify the conjecture?

$$
\begin{tikzcd}
 & \boxed{y_K} \arrow[dr] \arrow[ddd] & & \mathcal{O} \arrow[dl] & & \boxed{\rho_{\mathfrak{p}}} \arrow[dr] \arrow[drr] \arrow[ddd] & & \mathcal{L}_{\mathfrak{p}}(f,T) \arrow[dddl] & \mathrm{Sel}_{\mathfrak{p}} \arrow[dddl] \\
 & & \mathcal{O}y_K \arrow[d] & & & & H^1(G_i, A[\mathfrak{p}^i]) \arrow[ddl] & & \\
L'(f^{\alpha}/K,1) \arrow[dr] & & I_K \arrow[ddr] & & & & & & \\
 & \#Ш(A)_{\mathrm{an}} \arrow[ddrr] & & & & [\mathrm{KL}] \arrow[d] & & Ш(A)[\mathfrak{p}] \arrow[dl] & \\
 & & & & & \mathrm{BSD}(A) & & &
\end{tikzcd}
$$

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How to compute $\rho_{\mathfrak{p}}$?

# Which Galois representations are we interested in?

- Let $\mathfrak{p} \mid p$ be a regular prime ideal of $\mathcal{O}$ with residue field $\mathbf{F}_{\mathfrak{p}}$.
- We use the following Galois representations:
  - $\chi_p : \mathrm{Gal}(\overline{\mathbf{Q}}|\mathbf{Q}) \to \mathbf{F}_p^\times$ the mod-$p$ cyclotomic character,
  - the mod-$p$ Galois representation
    $\rho_p : \mathrm{Gal}(\overline{\mathbf{Q}}|\mathbf{Q}) \to \mathrm{Aut}_{\mathbf{Z}/p}(A[p](\overline{\mathbf{Q}})) \cong \mathrm{GL}_4(\mathbf{F}_p)$,
  - the mod-$\mathfrak{p}$ Galois representation
    $\rho_{\mathfrak{p}} : \mathrm{Gal}(\overline{\mathbf{Q}}|\mathbf{Q}) \to \mathrm{Aut}_{\mathcal{O}/\mathfrak{p}}(A[\mathfrak{p}](\overline{\mathbf{Q}})) \cong \mathrm{GL}_2(\mathbf{F}_{\mathfrak{p}})$.

# Which Galois representations are we interested in?

- Let $\mathfrak{p} \mid p$ be a regular prime ideal of $\mathcal{O}$ with residue field $\mathbf{F}_{\mathfrak{p}}$.
- We use the following Galois representations:
  - $\chi_p : \mathrm{Gal}(\overline{\mathbf{Q}}|\mathbf{Q}) \to \mathbf{F}_p^\times$ the mod-$p$ cyclotomic character,
  - the mod-$p$ Galois representation
    $\rho_p : \mathrm{Gal}(\overline{\mathbf{Q}}|\mathbf{Q}) \to \mathrm{Aut}_{\mathbf{Z}/p}(A[p](\overline{\mathbf{Q}})) \cong \mathrm{GL}_4(\mathbf{F}_p)$,
  - the mod-$\mathfrak{p}$ Galois representation
    $\rho_{\mathfrak{p}} : \mathrm{Gal}(\overline{\mathbf{Q}}|\mathbf{Q}) \to \mathrm{Aut}_{\mathcal{O}/\mathfrak{p}}(A[\mathfrak{p}](\overline{\mathbf{Q}})) \cong \mathrm{GL}_2(\mathbf{F}_{\mathfrak{p}})$.

# Excursion: Group theory of finite groups of Lie type

Assume $\mathfrak{p} \nmid 2$. Every subgroup of
$\mathrm{PSL}_2(\mathfrak{p}) := \mathbf{ker}\big(\overline{\det} : \mathrm{PGL}_2(\mathbf{F}_\mathfrak{p}) \twoheadrightarrow \mathbf{F}_\mathfrak{p}^\times/2\big)$ is (up to conjugacy)
contained in one of the following maximal subgroups:

- Borel subgroup $\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$ (reducible image as $\mathbf{F}_\mathfrak{p}$-representation)
- $\mathrm{PSL}_2(p)$ (reducible image as $\mathbf{F}_p$-representation, only for $\mathbf{F}_\mathfrak{p} \supsetneq \mathbf{F}_p$)
- normalizer of a split/non-split Cartan subgroup $C_s, C_{ns}$:
  $N_s = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \begin{pmatrix} 0 & b \\ a & 0 \end{pmatrix} \right\}; N_{ns} = \left\{ \begin{pmatrix} a & \varepsilon b \\ b & a \end{pmatrix}, \begin{pmatrix} a & \varepsilon b \\ -b & -a \end{pmatrix} \right\}$ (dihedral)
  with $\varepsilon \in \mathbf{F}_\mathfrak{p} \setminus \mathbf{F}_\mathfrak{p}^\square$
- $A_4$, $S_4$ or $A_5$ (exceptional image)

# How to prove almost all $\rho_{\mathfrak{p}}$ are irreducible?

- Raynaud's classification of simple factors of $\rho_p|_{I_p}^{\mathrm{ss}}$ and CFT:

  If $\rho_{\mathfrak{p}}$ is reducible, $\rho_{\mathfrak{p}}^{\mathrm{ss}} \cong \varepsilon \oplus \varepsilon^{-1}\chi_p$ with $\varepsilon$ of conductor $d$ with $d^2 \mid N$, or $\rho_{\mathfrak{p}}^{\mathrm{ss}} \cong \varepsilon\chi_p^n \oplus \varepsilon^{-1}\chi_p^{1-n}$ if $p^2 \mid N$.
- Hence: If $\rho_{\mathfrak{p}}$ is reducible as an $\mathbf{F}_{\mathfrak{p}}$-representation for all $\mathfrak{p} \mid p$, then

  an eigenvalue of $\rho_p(\mathrm{Frob}_\ell)$ has order dividing $\mathrm{ord}(\bar{\ell} \in (\mathbf{Z}/d)^\times)$.

  for $d = d_{\max}$ maximal such that $d_{\max}^2 \mid N$,
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- Hence: If $\rho_{\mathfrak{p}}$ is reducible as an $\mathbf{F}_{\mathfrak{p}}$-representation for all $\mathfrak{p} \mid p$, then

$$p \mid \mathrm{res}_{\mathbf{Z}[X]}\left(\mathrm{charpol}_{\mathbf{Z}[X]}(\rho_{p^\infty}(\mathrm{Frob}_\ell)), X^{\mathrm{ord}(\bar{\ell}\in(\mathbf{Z}/d)^\times)} - 1\right).$$

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$$p \mid \gcd_{\ell \nmid pN}\Big(\mathrm{res}_{\mathbf{Z}[X]}\left(\mathrm{charpol}_{\mathbf{Z}[X]}(\rho_{p^\infty}(\mathrm{Frob}_\ell)), X^{\mathrm{ord}(\bar{\ell}\in(\mathbf{Z}/d)^\times)} - 1\right)\Big).$$

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- Can be refined to $\mathfrak{p} \triangleleft \mathcal{O}$.

# How to prove almost all $\rho_{\mathfrak{p}}$ are maximal?

- If $\rho_{\mathfrak{p}}$ is irreducible as an $\mathbf{F}_{\mathfrak{p}}$-representation, but reducible as an $\mathbf{F}_p$-representation,

$$p \mid \gcd_{\substack{\ell \nmid pN \\ \ell \equiv 1 \pmod{d}}} \left( \frac{a_\ell - \overline{a_\ell}}{2} \right).$$

- If $p > 5$, $\rho_{\mathfrak{p}}$ does not have exceptional image.
- If $p > C(N, \deg \mathfrak{p})$, then the image of $\rho_{\mathfrak{p}}$ is not contained in the normalizer of a Cartan subgroup.

Questions?

# How to compute the Heegner index?

# How to compute a rational number exactly?

We can determine $x \in \mathbf{Q}$ exactly
if we know an explicit $N \in \mathbf{Z}$ such that $x \in \frac{1}{N}\mathbf{Z}$
and if we can approximate $x \in \mathbf{C}$ up to an arbitrary precision.
(We can determine $x \in \overline{\mathbf{Q}}$ exactly
if we know an explicit $N \in \mathbf{Z}$ such that $x \in \frac{1}{N}\overline{\mathbf{Z}}$ and $[\mathbf{Q}(x) : \mathbf{Q}] \leq d$
and if we can approximate $x \in \mathbf{C}$ up to an arbitrary precision.)

# How to go from the analytic rank to the algebraic rank?

Let $K/\mathbf{Q}$ be an imaginary quadratic Heegner field such that

- all primes $\ell \mid N$ split in $K$,
  i.e., there is a fractional ideal $\mathfrak{n} \triangleleft \mathcal{O}_K$ with $\mathcal{O}_K/\mathfrak{n} \cong \mathbf{Z}/N$,
- $L'(f/K, 1) \neq 0$.

(By results of Waldspurger et al., there are infinitely many such $K$.)

**Gross–Zagier formula, 1986**

For an isogeny quotient $\pi : J_0(N) \twoheadrightarrow A_f$ with Manin constant $c_\pi$ and Heegner point $y_K \in A_f(K)$,

$$L'(f/K, 1) = \frac{\|\omega_f\|^2}{c_\pi^2 u_K^2 \sqrt{|\operatorname{disc}_K|}} \hat{h}(y_K) \in \mathbf{R}_{\geq 0}.$$

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# How to compute the Heegner index $I_K$?

- We need to work on a Jacobian in the isogeny class of $A_f := J_0(N)/\mathrm{Ann}_{\mathbf{T}}(f)$.
- Compute an isogeny $\pi : A_f \to J := \mathrm{Jac}(X)$ using the big period matrices of $A_f$ and $J$.
- Compute the $h_K$ Heegner forms $Ax^2 + Bx + C$ for $(N, \mathrm{disc}_K)$.
- Compute the zeros $\tau$ of the Heegner forms with $\mathrm{Im}(\tau) > 0$.
- Approximate $\int_{q_\tau}^{i\infty} f(q)dq, \int_{q_\tau}^{i\infty} f^\alpha(q)dq$ with $q_\tau := \exp(2\pi i\tau)$ and map the points via $\pi : \mathbf{C}^2/\Lambda_f \to J(\mathbf{C})$ ($X$ hyperelliptic!) and sum over all $\tau$ to get a $\mathbf{C}$-approximation of $\pi(y_K)$.
- Find a $K$-approximation to $\pi(y_K) \in J(K)$.

Note that $\mathrm{Ann}_{\mathcal{O}}(J(K)/\mathcal{O}\pi(y_K))$ can be a multiple of $I_K := \mathrm{Ann}_{\mathcal{O}}(A_f(K) : \mathcal{O}y_K)$ if $[J(K) : \pi(A_f(K))] > 1$.

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# How to prove that our $\mathcal{O}\pi(y_K)$ is correct?

- Problem: The set of points of $J(K)$ near to some $y \in J(\mathbf{C})$ is dense!
- Idea: Use the Northcott property of heights $\hat{h}_{\mathscr{L}} : J(K) \to \mathbf{R}_{\geq 0}$ and assume that $\hat{h}_{2\vartheta}$ is injective on $J(K)$ up to sign and torsion.

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# How to compute $\#Ш(A/\mathbf{Q})_{\mathrm{an}}$ exactly?

- Compute $\frac{L(f,1)}{\Omega_f^+} \in \mathbf{Q}(f)$ exactly using modular symbols and Balakrishnan–Müller–Stein and van Bommel's code to compute $\Omega_A$.
- If $L(A,1) \neq 0$, this gives $\#Ш(A/\mathbf{Q})_{\mathrm{an}} \in \mathbf{Q}_{>0}$ exactly.
- If $L(A,1) = 0$:
  - Choose a Heegner field $K$ and compute $\frac{L(f_K,1)}{\mathrm{Reg}_{A/K}\,\Omega_{A/K}} \in \mathbf{Q}_{>0}$ exactly using Gross–Zagier, and hence compute $\#Ш(A/K)_{\mathrm{an}} \in \mathbf{Q}_{>0}$.
  - Compute $\#Ш(A^K/\mathbf{Q})_{\mathrm{an}} \in \mathbf{Q}_{>0}$ exactly.
  - Use $\#Ш(A/K)_{\mathrm{an}} = \#Ш(A/\mathbf{Q})_{\mathrm{an}} \cdot \#Ш(A^K/\mathbf{Q})_{\mathrm{an}}$ up to powers of 2 that can be explicitly bounded to compute $\#Ш(A/\mathbf{Q})_{\mathrm{an}} \in \mathbf{Q}_{>0}$ exactly.

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Questions?

# Examples in dimension 2

# Can you give an example? $A = \mathrm{Jac}(X_0(39)/w_{13})$

- $\mathcal{O} = \mathbf{Z}[\sqrt{2}]$
- $r = r_{\mathrm{an}} = 0$
- $\#Ш(A/\mathbf{Q})_{\mathrm{an}} = 1$
- $A(\mathbf{Q}) = A(\mathbf{Q})_{\mathrm{tors}} \cong \mathbf{Z}/2 \times \mathbf{Z}/(2 \cdot 7)$
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# All Atkin-Lehner quotients of genus 2 of our type (I)

| $X$ | $r$ | $\mathcal{O}$ | $\#Ш_{an}$ | $\rho_{\mathfrak{p}}$ red. | $c$ | $(D, I_D)$ | $\#Ш$ |
|---|---|---|---|---|---|---|---|
| $X_0(23)$ | 0 | $\sqrt{5}$ | 1 | $11_1$ | 11 | $(-7, 11)$ | $11^0$ |
| $X_0(29)$ | 0 | $\sqrt{2}$ | 1 | $7_1$ | 7 | $(-7, 7)$ | $7^0$ |
| $X_0(31)$ | 0 | $\sqrt{5}$ | 1 | $\sqrt{5}$ | 5 | $(-11, 5)$ | $5^0$ |
| $X_0(35)/w_7$ | 0 | $\sqrt{17}$ | 1 | $2_1$ | 1 | $(-19, 1)$ | 1 |
| $X_0(39)/w_{13}$ | 0 | $\sqrt{2}$ | 1 | $\sqrt{2}, 7_1$ | 7 | $(-23, 7)$ | $7^0$ |
| $X_0(67)^+$ | 2 | $\sqrt{5}$ | 1 | | 1 | $(-7, 1)$ | 1 |
| $X_0(73)^+$ | 2 | $\sqrt{5}$ | 1 | | 1 | $(-19, 1)$ | 1 |
| $X_0(85)^*$ | 2 | $\sqrt{2}$ | 1 | $\sqrt{2}$ | 1 | $(-19, 1)$ | 1 |
| $X_0(87)/w_{29}$ | 0 | $\sqrt{5}$ | 1 | $\sqrt{5}$ | 5 | $(-23, 5)$ | $5^0$ |
| $X_0(93)^*$ | 2 | $\sqrt{5}$ | 1 | | 1 | $(-11, 1)$ | 1 |
| $X_0(103)^+$ | 2 | $\sqrt{5}$ | 1 | | 1 | $(-11, 1)$ | 1 |
| $X_0(107)^+$ | 2 | $\sqrt{5}$ | 1 | | 1 | $(-7, 1)$ | 1 |
| $X_0(115)^*$ | 2 | $\sqrt{5}$ | 1 | | 1 | $(-11, 1)$ | 1 |
| $X_0(125)^+$ | 2 | $\sqrt{5}$ | 1 | $\sqrt{5}$ | 1 | $(-11, 1)$ | $5^0$ |

# All Atkin-Lehner quotients of genus 2 of our type (II)

| $X$ | $r$ | $\mathcal{O}$ | $\#Ш_{an}$ | $\rho_{\mathfrak{p}}$ red. | $c$ | $(D, I_D)$ | $\#Ш$ |
|---|---|---|---|---|---|---|---|
| $X_0(133)^*$ | 2 | $\sqrt{5}$ | 1 | | 1 | $(-31, 1)$ | 1 |
| $X_0(147)^*$ | 2 | $\sqrt{2}$ | 1 | $\sqrt{2}, 7_1$ | 1 | $(-47, 1)$ | $7^0$ |
| $X_0(161)^*$ | 2 | $\sqrt{5}$ | 1 | | 1 | $(-19, 1)$ | 1 |
| $X_0(165)^*$ | 2 | $\sqrt{2}$ | 1 | $\sqrt{2}$ | 1 | $(-131, 1)$ | 1 |
| $X_0(167)^+$ | 2 | $\sqrt{5}$ | 1 | | 1 | $(-15, 1)$ | 1 |
| $X_0(177)^*$ | 2 | $\sqrt{5}$ | 1 | | 1 | $(-11, 1)$ | 1 |
| $X_0(191)^+$ | 2 | $\sqrt{5}$ | 1 | | 1 | $(-7, 1)$ | 1 |
| $X_0(205)^*$ | 2 | $\sqrt{5}$ | 1 | | 1 | $(-31, 1)$ | 1 |
| $X_0(209)^*$ | 2 | $\sqrt{2}$ | 1 | | 1 | $(-51, 1)$ | 1 |
| $X_0(213)^*$ | 2 | $\sqrt{5}$ | 1 | | 1 | $(-11, 1)$ | 1 |
| $X_0(221)^*$ | 2 | $\sqrt{5}$ | 1 | | 1 | $(-35, 1)$ | 1 |
| $X_0(287)^*$ | 2 | $\sqrt{5}$ | 1 | | 1 | $(-31, 1)$ | 1 |
| $X_0(299)^*$ | 2 | $\sqrt{5}$ | 1 | | 1 | $(-43, 1)$ | 1 |
| $X_0(357)^*$ | 2 | $\sqrt{2}$ | 1 | | 1 | $(-47, 1)$ | 1 |

Questions?

# What is left to do?

# What would be nice to achieve?

- Using Shnidman–Weiss[2], find examples of $A/\mathbf{Q}$ with

$$\#Ш(A/\mathbf{Q}) = \#Ш(A/\mathbf{Q})_{\mathrm{an}} \neq 2^i!$$

  Can have $p \in \{3, 5, 7, 11, (13?), \ldots, (31?), \ldots ?\}$.
- Find $J/\mathbf{Q}$ and $\mathfrak{p} \mid p$ "large" with
  - $p^2 \mid N$ (no $p$-adic $L$-functions),
  - $\mathfrak{p} \mid c \cdot I_K$ ([KL] does not give $Ш(J/\mathbf{Q})[\mathfrak{p}] = 0$), and
  - $\rho_{\mathfrak{p}}$ irreducible ($\mathfrak{p}$-descent hard)!

[2] *Elements of prime order in Tate-Shafarevich groups of abelian varieties over $\mathbb{Q}$*, `arXiv:2106.14096`

# What are the next steps and projects?

- Almost done: Verification for all 97 genus 2 curves with absolutely simple modular Jacobian from the LMFDB.
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Thank you!