KLEINE AG: EXISTENCE OF MODIFICATIONS

Organization:
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A modification of algebraic spaces is a pair consisting of a proper morphism of algebraic spaces

$$f : X' \to X$$

and a closed subspace $Y' \subset X'$ such that f is an isomorphism away from Y'.

In this kleine AG, we will study Artin's results on the existence of such modifications. More precisely, Artin proves in [Art2] (as an application of [Art1]) that such modifications exist whenever they exist formally and we will restrict ourselves to the study of contractions.

This is motivated by an older result of Grauert (see [Gra]) stating that negative definite, complete, connected curves on complex surfaces are always contractible in the category of complex spaces.

The rough idea of this program is as follows:

First, accepting the results of Grauert, we show that existence of modifications is not true in the category of schemes. Therefore, we introduce algebraic spaces and reformulate the problem of finding a contraction realizing a given formal contraction as a modular problem, i.e. as the problem of representing a suitable functor. Then we solve the modular problem and, ideally, apply the results to the example given in the first talk.

The existence of modifications has applications to the theory of algebraic surfaces, generalizing the contraction results of Grauert [Gra] to positive characteristic, as well as Moishezon spaces.

Talk 1: An example in the category of schemes. (30 minutes)

Define contractions of schemes (just replace algebraic space by scheme in [Art2][p. 88]). State the result of Grauert concerning existence of contractions of curves with negative self-intersection on complex surfaces (last statement in [Gra]). Show that this result does not hold in the category of schemes by explaining Example 3.1 in [Bad].

Talk 2: Introduction to Algebraic Spaces. (60 minutes)

Define the étale site on the category of schemes (e.g. [Knut][Chapter I.4] or [St]). Define étale equivalence relations and give the two equivalent definitions of (quasiseparated) Algebraic Spaces given in [Knut][p.92-93]. Explain how to define properties (which are stable in the étale topology) of morphisms of algebraic spaces using a representable étale covering [Knut][p.95,p.106 ff.].

Give the definition of a formal algebraic space. [Knut][p.215]

Talk 3: The Contraction Functor. (45-60 minutes)

Define formal contractions of algebraic spaces ([Art2][p.90-92]). Re-phrase the problem of finding a contraction as a modular problem ([Art2][p.100-101]) and, if time permits, prove the equivalence of these problems (Lemma 3.7 in [Art2]). In any case, mention Lemma 3.7.

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Applying Artin's representability criterion. The following two talks are the technical heart of this kleine AG. However, the proof of existence of contractions is too long to be presented with all details in these talks. Therefore, although the description of these talks wants the speakers to state and prove all conditions of Artin's representability criterion, the speakers of these talks should coordinate to only state and prove a selection of the conditions.

Talk 4: A criterion for representability. (45 minutes)

Explain the conditions and give the statements of Theorem 3.4 and Theorem 3.7 of [Art1] for a set-valued functor to be represented by an algebraic space. State the completely formal extension of these results (Proposition 3.11 of [Art2]).

Talk 5: Proof of representability. (45 minutes) State Theorem 3.1 of [Art2], i.e. existence of contractions, and prove it by applying the representability criterion of the previous talk. This is chapter 4 of [Art2].

Talk 6 (optional): Applications. (30 minutes) Sketch the proof of Theorem 6.2 of [Art2] and, using Corollary 6.12 of [Art2], deduce that the curve of talk 1 can be contracted in the category of algebraic spaces.

References

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