On an analogue of the conjecture of Birch and Swinnerton-Dyer for Abelian schemes over higher dimensional bases over finite fields – complements

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Abstract

We prove a finiteness theorem for the first flat cohomology group of finite flat group schemes over integral normal proper varieties over finite fields. As a consequence, we can prove the invariance of the finiteness of the Tate-Shafarevich group of Abelian schemes over higher dimensional bases under isogenies and alterations over/of such bases for the p-part. Along the way, we generalize previous results on the Tate-Shafarevich and the Brauer group in this situation.

Keywords: Abelian varieties of dimension > 1; Birch-Swinnerton-Dyer conjecture; Étale and other Grothendieck topologies and cohomologies; Brauer groups of schemes; Arithmetic ground fields

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1 Introduction

The Tate-Shafarevich group $Ш(\mathscr{A}/X)$ of an Abelian scheme $\mathscr{A}$ over a base scheme X is of great importance for the arithmetic of $\mathscr{A}$. It classifies everywhere locally trivial $\mathscr{A}$-torsors. Its finiteness is sufficient to establish our analogue of the conjecture of Birch and Swinnerton-Dyer [Kel19] over higher dimensional bases over finite fields.

In [Kel16, section 4.3], we showed that finiteness of an ℓ-primary component of the Tate-Shafarevich group descends under generically étale alterations of generical degree prime to ℓ for ℓ invertible on the base scheme. This is used in [Kel19, Corollary 5.11] to prove the finiteness of the Tate-Shafarevich group and an analogue of the Birch-Swinnerton-Dyer conjecture for certain Abelian schemes over higher dimensional bases over finite fields under mild conditions. In [Kel16, section 4.4], we showed that finiteness of an ℓ-primary component of the Tate-Shafarevich group is invariant under étale isogenies. In this article, we prove these results also for the p^∞-torsion.

Recall that we defined the Tate-Shafarevich group of $\mathscr{A}/X$ for $\dim X > 1$ and $\mathscr{A}$ of good reduction as $\mathrm{H}^1_{\text{ét}}(X, \mathscr{A})$. In [Kel16, Lemma 4.15] we proved as a hypothesis in [Kel16, Theorem 4.5]:

Lemma 1.1. *Let X/k be a smooth variety and $\mathscr{C}/X$ a smooth proper relative curve. Assume $\dim X \leq 2$. Let $Z \hookrightarrow X$ be a reduced closed subscheme of codimension ≥ 2. Then*

$$\mathrm{H}^i_Z(X, \mathbf{Pic}^0_{\mathscr{C}/X}) = 0 \quad \text{for } i \leq 2.$$

If $\dim X > 2$, this holds at least up to p-torsion.

We weaken the hypothesis that $\mathscr{A}$ is a Jacobian in [Kel16, Lemma 4.15] to arbitrary Abelian schemes:

Theorem (Corollary 2.5)**.** *Let X be a regular integral Noetherian separated scheme and $\mathscr{A}/X$ be an Abelian scheme. Let $Z \hookrightarrow X$ be a closed subscheme of codimension ≥ 2. Then the vanishing condition [Kel16, (4.4)] holds for $\mathscr{A}/X$: $\mathrm{H}^i_Z(X, \mathscr{A})$ is torsion for all i. Furthermore, $\mathrm{H}^0_Z(X, \mathscr{A}) = 0$, and for $i = 1, 2$, the only possible torsion is p-torsion for p not invertible on X.*

Our main results are now as follows:

Theorem (Theorem 3.14)**.** *Let X be a proper integral normal variety over a finite field and G/X be a finite flat commutative group scheme. Then $\mathrm{H}^1_{\mathrm{fppf}}(X,G)$ is finite.*

This theorem is proven by reduction to the finite flat simple group schemes $\mathbf{Z}/p$,μ_p and α_p over an algebraically closed field using de Jong's alteration theorem, Raynaud-Gruson and a dévissage argument.

Using this technical result and refining our methods from [Kel16], we obtain the following three results:

The following theorem has been proved as [Kel16, Lemma 4.28] for p prime to the characteristic of k; in this article, we prove it also for p equal to the characteristic of k:

Theorem (Lemma 5.1)**.** *Let $\mathscr{A}/X$ be an Abelian scheme over a proper variety X over a finite field of characteristic p. Then $Ш(\mathscr{A}/X)[p^\infty]$ is cofinitely generated.*

In [Kel16, Theorem 4.31], we proved:

Theorem 1.2. *Let X/k be proper, $\mathscr{A}$ and $\mathscr{A}'$ Abelian schemes a variety X over a finite field and $f:\mathscr{A}'\to\mathscr{A}$ an étale isogeny. Let $\ell\neq\operatorname{char}k$ be a prime. Then $Ш(\mathscr{A}/X)[\ell^\infty]$ is finite if and only if $Ш(\mathscr{A}'/X)[\ell^\infty]$ is finite.*

In this article, we prove it also for ℓ equal to the characteristic of k:

Theorem (invariance of finiteness of Ш under isogenies, Theorem 4.1)**.** *Let X/k be a proper variety over a finite field k and $f:\mathscr{A}\to\mathscr{A}'$ be an isogeny of Abelian schemes over X. Let p be an arbitrary prime. Assume f étale if $p\neq\operatorname{char}k$. Then $Ш(\mathscr{A}/X)[p^\infty]$ is finite if and only if $Ш(\mathscr{A}'/X)[p^\infty]$ is finite.*

In [Kel16, Theorem 4.29], we proved:

Theorem 1.3. *Let $f:X'\to X$ be a morphism of normal integral varieties over a finite field which is an alteration of degree prime to ℓ for a prime ℓ invertible on X, i.e., f is a proper, surjective, generically étale morphism of generical degree prime to ℓ. If $\mathscr{A}$ is an Abelian scheme on X such that the ℓ^∞-torsion of the Tate-Shafarevich group $Ш(\mathscr{A}'/X')$ of $\mathscr{A}':=f^*\mathscr{A}=\mathscr{A}\times_X X'$ is finite, then the ℓ^∞-torsion of the Tate-Shafarevich group $Ш(\mathscr{A}/X)$ is finite.*

In this article, we prove it also for ℓ equal to the characteristic of k and remove the condition that the generical degree is prime to ℓ if ℓ is invertible on X:

Theorem (invariance of finiteness of Ш under alterations, Theorem 5.3 and Theorem 5.5)**.** *Let $f:X'\to X$ be a proper, surjective, generically finite morphism of generical degree d of regular, integral, separated varieties over a finite field of characteristic $p>0$. Let $\mathscr{A}$ be an abelian scheme on X and $\mathscr{A}':=f^*\mathscr{A}=\mathscr{A}\times_X X'$. Let ℓ be an arbitrary prime. Assume $(d,\ell)=1$ if $\ell=p$. If $Ш(\mathscr{A}'/X')[\ell^\infty]$ is finite, so is $Ш(\mathscr{A}/X)[\ell^\infty]$.*

Notation. Canonical isomorphisms are often denoted by "=".

We denote Pontrjagin duality by $(-)^D$ and duals of Abelian schemes and Cartier duals by $(-)^t$.

For a scheme X, we denote the set of codimension-1 points by $X^{(i)}$ and the set of closed points by $|X|$.

For an abelian group A, let A_{tors} be the torsion subgroup of A, and $A_{\text{n-tors}}=A/A_{\mathrm{tors}}$. For A a cofinitely generated ℓ-primary group, let A_{div} be the maximal divisible subgroup of A, which equals the subgroup of divisible elements of A in this case ([Kel19, Lemma 2.1.1 (iii)]), and $A_{\text{n-div}}=A/A_{\mathrm{div}}$. For an integer n and an object A of an abelian category, denote the cokernel of $A\xrightarrow{n}A$ by A/n and its kernel by $A[n]$, and for a prime p the p-primary subgroup $\varinjlim_n A[p^n]$ by $A[p^\infty]$. Write $A[\text{non-}p]$ or $A[p']$ for $\varinjlim_{p\nmid n}A[n]$. For a prime ℓ, let the ℓ-adic Tate module $T_\ell A$ be $\varprojlim_n A[\ell^n]$ and the rationalized ℓ-adic Tate module $V_\ell A=T_\ell A\otimes_{\mathbf{Z}_\ell}\mathbf{Q}_\ell$. The corank of $A[p^\infty]$ is the $\mathbf{Z}_p$-rank of $A[p^\infty]^D=T_pA$.

2 Vanishing of étale cohomology with supports of Abelian schemes

This is a complement to the "vanishing condition" $\mathrm{H}^i_Z(X,G)=0$ [Kel16, (4.4)], which is proven there only for Jacobians of curves, see [Kel16, Lemma 4.10].

Theorem 2.1. *Let X be a regular integral Noetherian separated scheme and G/X be a finite étale commutative group scheme of order invertible on X. Let $Z\hookrightarrow X$ be a closed subscheme of codimension ≥ 2. Then $\mathrm{H}^i_Z(X,G)=0$ for $i\leq 2$ (étale cohomology with supports in Z).*

Proof. Let $U = X \setminus Z$. One has a long exact cohomology sequence

$$\ldots \to \mathrm{H}^{i-1}(X,G) \to \mathrm{H}^{i-1}(U,G) \to \mathrm{H}^i_Z(X,G) \to \mathrm{H}^i(X,G) \to \mathrm{H}^i(U,G) \to \ldots,$$

so one has to prove that $\mathrm{H}^i(X,G) \to \mathrm{H}^i(U,G)$ is an isomorphism for $i = 0, 1$ and injective for $i = 2$.

For $i = 0$, the claim $\mathrm{H}^i_Z(X,G) = 0$ is equivalent to the injectivity of

$$\mathrm{H}^0(X,G) \to \mathrm{H}^0(U,G),$$

which is clear from [Har83, p. 105, Exercise II.4.2] since G/X is separated, X is reduced and $U \hookrightarrow X$ is dense.

For $i = 1$ the claim $\mathrm{H}^i_Z(X,G) = 0$ is equivalent to

$$\mathrm{H}^0(X,G) \to \mathrm{H}^0(U,G)$$

being surjective and

$$\mathrm{H}^1(X,G) \to \mathrm{H}^1(U,G)$$

being injective. The surjectivity of $\mathrm{H}^0(X,G) \to \mathrm{H}^0(U,G)$ follows e. g. from

Theorem 2.2. *Let S be a normal Noetherian base scheme, and let $u : T \dashrightarrow G$ be an S-rational map from a smooth S-scheme T to a smooth and separated S-group scheme G. Then, if u is defined in codimension ≤ 1, it is defined everywhere.*

Proof. See [BLR90, p. 109, Theorem 1]. □

For the injectivity of $\mathrm{H}^1(X,G) \to \mathrm{H}^1(U,G)$: If a principal homogeneous space P/X for G/X is trivial over U, then it is trivial over X: The trivialization over U gives a rational map from X to the principal homogeneous space and any such map (with X a regular scheme) extends to a morphism by Theorem 2.2.

For the surjectivity of $\mathrm{H}^1(X,G) \to \mathrm{H}^1(U,G)$: This means that any principal homogeneous space P/U extends to a principal homogeneous space $\bar{P}/X$. By [Mil80, p. 123, Corollary III.4.7], we have $\mathrm{PHS}(G/X) \xrightarrow{\sim} \check{\mathrm{H}}^1(X_{\mathrm{fl}}, G)$ (Čech cohomology) since G/X is affine. Since G/X is smooth, [Mil80, p. 123, Remark III.4.8 (a)] shows that we can take étale cohomology as well, and by [Mil80, p. 101, Corollary III.2.10], one can take derived functor cohomology instead of Čech cohomology. Recall:

Theorem 2.3 (Zariski-Nagata purity)**.** *Let X be a locally Noetherian regular scheme and U an open subscheme with closed complement of codimension ≥ 2. Then the functor $X' \mapsto X' \times_X U$ is an equivalence of categories from étale coverings of X to étale coverings of U.*

Proof. See [SGA1, Exp. X, Corollaire 3.3]. □

By Theorem 2.3, one can extend P/U uniquely to a $\bar{P}/X$, for which we have to show that it represents an element of $\mathrm{H}^1(X,G)$, i. e., that it is a G-torsor.

So we need to show that if P/U is an $G|_U$-torsor and $\bar{P}$ an extension of P to a finite étale covering of X, then $\bar{P}/X$ is also an G-torsor. For this, we use the following

Theorem 2.4. *Let X be a connected scheme, $G \to X$ a finite flat group scheme, and $\bar{P} \to X$ a scheme over X equipped with a left action $\rho : G \times_X \bar{P} \to \bar{P}$. These data define a G-torsor over X if and only if there exists a finite locally free surjective morphism $Y \to X$ such that $\bar{P} \times_X Y \to Y$ is isomorphic, as a Y-scheme with $G \times_X Y$-action, to $G \times_X Y$ acting on itself by left translations.*

Proof. See [Sza09, p. 171, Lemma 5.3.13]. □

That P/U is an $G|_U$-torsor amounts to saying that there is an operation

$$G|_U \times_U P \to P$$

as in the previous Theorem 2.4. Since this is étale locally isomorphic to the canonical action

$$G|_U \times_U G|_U \stackrel{\mu}{\to} G|_U$$

which is finite étale, by faithfully flat descent the operation defines an étale covering, so extends by Zariski-Nagata purity (Theorem 2.3) uniquely to an étale covering $H \to X$, which by uniqueness has to be isomorphic to $G \times_X \bar{P} \to \bar{P}$. Now a routine check shows the condition in Theorem 2.4.

There is a finite étale Galois covering X'/X with Galois group G such that $G \times_X X'$ is isomorphic to a direct sum of μ_n with n invertible on X. The Leray spectral sequence with supports $\mathrm{H}^p(G, \mathrm{H}^q_{Z'}(X', G \times_X X')) \Rightarrow \mathrm{H}^{p+q}_Z(X, G)$ from [Kel16, p. 228, Theorem 4.9], so it suffices to show $\mathrm{H}^q_Z(X', G \times_X X') = 0$ for $q = 0, 1, 2$. Hence one can assume $G \cong \mu_n$ for n invertible on X.

By [Mil80, Example III.2.22], one has an injection $\mathrm{Br}(X) \hookrightarrow \mathrm{Br}(K(X))$ with $K(X)$ the function field of X and $\mathrm{Br}(X) \to \mathrm{Br}(U) \to \mathrm{Br}(K(X))$, so $\mathrm{Br}(X) \to \mathrm{Br}(U)$ is injective. By the hypotheses on X and since the codimension of Z in X is ≥ 2, there is a restriction isomorphism $\mathrm{Pic}(X) \xrightarrow{\sim} \mathrm{Pic}(U)$ (because of the codimension condition and [Har83, Proposition II.6.5 (b)], $\mathrm{Cl}\, X \xrightarrow{\sim} \mathrm{Cl}\, U$, and because of [Har83, Proposition II.6.16], $\mathrm{Cl}\, X \cong \mathrm{Pic}\, X$ functorial in the scheme). Hence the snake lemma applied to the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Pic}(X)/n & \longrightarrow & \mathrm{H}^2(X, \mu_n) & \longrightarrow & \mathrm{Br}(X)[n] & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{Pic}(U)/n & \longrightarrow & \mathrm{H}^2(U, \mu_n) & \longrightarrow & \mathrm{Br}(U)[n] & \longrightarrow & 0 \end{array}$$

gives that $\mathrm{H}^2(X, \mu_n) \to \mathrm{H}^2(U, \mu_n)$ is injective, so $\mathrm{H}^2_Z(X, \mu_n) = 0$. □

Corollary 2.5. *Let X be a regular integral Noetherian separated scheme and $\mathscr{A}/X$ be an Abelian scheme. Let $Z \hookrightarrow X$ be a closed subscheme of codimension ≥ 2. Then $\mathrm{H}^i_Z(X, \mathscr{A})$ is torsion for all i. Furthermore, $\mathrm{H}^0_Z(X, \mathscr{A}) = 0$, and for $i = 1, 2$, the only possible torsion is p-torsion for p not invertible on X.*

Proof. By [Kel16, p. 224, Proposition 4.1], $\mathrm{H}^i(X, \mathscr{A})$ is torsion for $i > 0$. The Kummer exact sequence $0 \to \mathscr{A}[n] \to \mathscr{A} \to \mathscr{A} \to 0$ for n invertible on X yields a surjection

$$\mathrm{H}^i_Z(X, \mathscr{A}[n]) \twoheadrightarrow \mathrm{H}^i_Z(X, \mathscr{A})[n],$$

so it suffices to show that $\mathrm{H}^i_Z(X, \mathscr{A}[n]) = 0$ for $i = 1, 2$. But this is Theorem 2.1. The triviality $\mathrm{H}^0_Z(X, \mathscr{A}) = 0$ is equivalent to the injectivity of

$$\mathrm{H}^0(X, \mathscr{A}) \to \mathrm{H}^0(U, \mathscr{A}),$$

which is clear from [Har83, p. 105, Exercise II.4.2] since $\mathscr{A}/X$ is separated, X is reduced and $U \hookrightarrow X$ is dense. □

With vanishing condition (4.4) in [Kel16, Theorem 4.5] satisfied for $\mathscr{A}/X$ by Corollary 2.5, the statement there generalizes from $\mathscr{A}$ a Jacobian to $\mathscr{A}$ a general Abelian scheme:

Theorem 2.6. *Let X be regular, Noetherian, integral and separated and let $\mathscr{A}$ be an Abelian scheme over X. For $x \in X$, denote the function field of X by K, the quotient field of the strict Henselization of $\mathscr{O}_{X,x}$ by K_x^{nr}, the inclusion of the generic point by $j : \{\eta\} \hookrightarrow X$ and let $j_x : \mathrm{Spec}(K_x^{nr}) \hookrightarrow \mathrm{Spec}(\mathscr{O}_{X,x}^{sh}) \hookrightarrow X$ be the composition. Then we have*

$$\mathrm{H}^1(X, \mathscr{A}) \xrightarrow{\sim} \ker\Big(\mathrm{H}^1(K, j^*\mathscr{A}) \to \prod_{x \in X} \mathrm{H}^1(K_x^{nr}, j_x^*\mathscr{A})\Big).$$

One can replace the product over all points by the following:
(a) the closed points $x \in |X|$: One has isomorphisms

$$\mathrm{H}^1(X, \mathscr{A}) \xrightarrow{\sim} \ker\Big(\mathrm{H}^1(K, j^*\mathscr{A}) \to \prod_{x \in |X|} \mathrm{H}^1(K_x^{nr}, j_x^*\mathscr{A})\Big)$$

and

$$\mathrm{H}^1(X, \mathscr{A}) \xrightarrow{\sim} \ker\Big(\mathrm{H}^1(K, j^*\mathscr{A}) \to \prod_{x \in |X|} \mathrm{H}^1(K_x^{h}, j_x^*\mathscr{A})\Big)$$

with $K_x^h = \mathrm{Quot}(\mathscr{O}_{X,x}^h)$ the quotient field of the Henselization if $\kappa(x)$ is finite.

or (b) the codimension-1 points $x \in X^{(1)}$: One has an isomorphism

$$\mathrm{H}^1(X, \mathscr{A}) \xrightarrow{\sim} \ker\Big(\mathrm{H}^1(K, j^*\mathscr{A}) \to \bigoplus_{x \in X^{(1)}} \mathrm{H}^1(K_x^{nr}, j_x^*\mathscr{A})\Big)$$

if one disregards the p-torsion ($p = \operatorname{char} k$) and X/k is smooth projective over k finitely generated. For $\dim X \leq 2$, this also holds for the p-torsion.

For $x \in X^{(1)}$, one can also replace K_x^{nr} and K_x^h by the quotient field of the completions $\hat{\mathscr{O}}^{sh}_{X,x}$ and $\hat{\mathscr{O}}^{h}_{X,x}$, respectively.

3 Finiteness theorems for $\mathrm{H}^1_{\mathrm{fppf}}$ over finite fields

The aim of this section is to show that $\mathrm{H}^1_{\mathrm{fppf}}(X, G)$ is finite for X a normal proper variety over a finite field of characteristic p and G/X a finite flat group scheme.

The proof is by reduction to the case of a finite flat *simple* group scheme over an algebraically closed field, which is isomorphic to $\mathbf{Z}/\ell$ (étale-étale), $\mathbf{Z}/p$ (étale-local), μ_p (local-étale) or α_p (local-local).

We use the interpretation of $\mathrm{H}^1_{\mathrm{fppf}}(X, G)$ as G-torsors on X [Mil80, Proposition III.4.7] since G/X is affine. We also exploit de Jong's alteration theorem [de 96, Theorem 4.1].

Let us first recall some well-known facts on flat cohomology.

Definition 3.1. *An* **isogeny** *of commutative group schemes G, H of finite type over an arbitrary base scheme X is a group scheme homomorphism $f : G \to H$ such that for all $x \in X$, the induced homomorphism $f_x : G_x \to H_x$ on the fibers over x is finite and surjective on identity components.*

Remark 3.2. See [BLR90, p. 180, Definition 4]. We will usually consider isogenies between abelian schemes, for example the finite flat n-multiplication, which is étale iff n is invertible on the base scheme or the abelian schemes are trivial.

Lemma 3.3. *Let G, G' be commutative group schemes over a scheme X which are smooth and of finite type over X with connected fibers and $\dim G = \dim G'$ and let $f : G' \to G$ be a morphism of commutative group schemes over X.*

If f is flat (respectively, étale) then $\ker(f)$ is a flat (respectively, étale) group scheme over X, f is quasi-finite, surjective and defines an epimorphism in the category of flat (respectively, étale) sheaves over X.

Proof. See [Kel19, Lemma 2.3.3]. ☐

Lemma 3.4 (Kummer sequence)**.** *Let $f : G \to G'$ be a faithfully flat isogeny between smooth commutative group schemes over a base scheme X. Then the sequence*

$$0 \to \ker(f) \to G \xrightarrow{f} G' \to 0$$

is exact on X_{fppf}. This applies in particular to $G = \mathbf{G}_m$ and $G = \mathscr{A}$ an abelian scheme and the n-multiplication morphism for arbitrary $n \neq 0$.

Proof. Since f is faithfully flat, in particular surjective, it is an epimorphism of sheaves by Lemma 3.3. An isogeny of abelian schemes is faithfully flat by [Mil86, Proposition 8.1]. ☐

Lemma 3.5. *Let G/X be a* smooth *commutative group scheme. Then there are comparison isomorphisms*

$$\mathrm{H}^i_{\mathrm{fppf}}(X, G) = \mathrm{H}^i_{\mathrm{\acute{e}t}}(X, G).$$

In particular, $\mathrm{H}^i_{\mathrm{fppf}}(X, G)$ is finite if X is proper over a finite field and G is a commutative finite étale group scheme.

Proof. See [Mil80, Remark III.3.11 (b)] and note that the proof given there gives a comparison isomorphism for any topologies between the étale and the flat site. ☐

Lemma 3.6. *Let X be a Noetherian integral scheme with function field $K(X)$ and $U \subseteq X$ dense open. Then there is an exact sequence*

$$1 \to \mathbf{G}_m(X) \to \mathbf{G}_m(U) \to \bigoplus_{D \in (X \setminus U)^{(1)}} \mathbf{Z}[D] \to \mathrm{Cl}(X) \to \mathrm{Cl}(U) \to 0.$$

Proof. The assumptions imply that there is a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathscr{O}_X(X)^\times & \longrightarrow & K(X)^\times & \longrightarrow & \mathrm{Div}(X) & \longrightarrow & \mathrm{Cl}(X) & \longrightarrow & 0 \\ & & \downarrow & & \| & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mathscr{O}_X(U)^\times & \longrightarrow & K(U)^\times & \longrightarrow & \mathrm{Div}(U) & \longrightarrow & \mathrm{Cl}(U) & \longrightarrow & 0. \end{array}$$

A diagram chase yields the result. □

Corollary 3.7. *Let X be a Noetherian integral regular scheme and let $U \subseteq X$ be dense open. Then there is an exact sequence*

$$1 \to \mathbf{G}_m(X) \to \mathbf{G}_m(U) \to \bigoplus_{D \in (X \setminus U)^{(1)}} \mathbf{Z}[D] \to \mathrm{Pic}(X) \to \mathrm{Pic}(U) \to 0.$$

Proof. By the assumptions, $\mathrm{Cl}(X) = \mathrm{Pic}(X)$ and $\mathrm{Cl}(U) = \mathrm{Pic}(U)$. □

Corollary 3.8. *Let $X/\mathbf{F}_q$ be an integral Noetherian regular proper variety and let $j : U \hookrightarrow X$ be the inclusion of an open subscheme of X. Then $\mathrm{H}^1_{\mathrm{fppf}}(U, \mu_{p^n})$ is finite for all n and any prime p.*

Proof. The Kummer sequence Lemma 3.4 on U_{fppf} together with $\mathrm{Pic}(U) = \mathrm{H}^1_{\mathrm{fppf}}(U, \mathbf{G}_{m,U})$ by Lemma 3.5 yields the exact sequence

$$1 \to \mathbf{G}_m(U)/p^n \to \mathrm{H}^1_{\mathrm{fppf}}(U, \mu_{p^n}) \to \mathrm{Pic}(U)[p^n] \to 0.$$

Since $\mathbf{G}_m(X) = \Gamma(X, \mathbf{G}_m)^\times$ is finite by the coherence theorem [EGAIII$_1$, Thm. (3.2.1)], since $X/\mathbf{F}_q$ is proper and $\mathbf{F}_q$ is finite, and since $\mathrm{Pic}(X)$ is finitely generated since its sits in a short exact sequence $0 \to \mathrm{Pic}^0(X) \to \mathrm{Pic}(X) \to \mathrm{NS}(X) \to 0$ and $\mathrm{Pic}^0(X)$ is finite since it is the group of rational points of an Abelian variety over a finite field and $\mathrm{NS}(X)$ is always finitely generated by [SGA6, Exp. XIII, § 5], by Corollary 3.7 and the finiteness of $(X \setminus U)^{(1)}$, this exact sequence gives the finiteness of $\mathbf{G}_m(U)/p^n$ and of $\mathrm{Pic}(U)[p^n]$. □

The following statements and proofs in this section are an extended version of the sketch of Theorem 3.14 given by 'darx' in [MO257441].

Lemma 3.9. *Let X be a normal integral scheme and G/X be a finite flat group scheme. If T is a G-torsor on X trivial over the generic point of X, then T is trivial. Hence, $\mathrm{H}^1_{\mathrm{fppf}}(X, G) \to \mathrm{H}^1_{\mathrm{fppf}}(K(X), G)$ is injective, and if $f : Y \to X$ is birational, $f^* : \mathrm{H}^1_{\mathrm{fppf}}(X, G) \to \mathrm{H}^1_{\mathrm{fppf}}(Y, G)$ is injective.*

Proof. Since T is trivial over the generic point of X, generically, there is a section of $\pi : T \to X$. This extends to a rational map $\sigma : X \dashrightarrow T$. Take the schematic closure $i : X' \hookrightarrow T$ of σ. The composition $\pi \circ i : X' \to T \to X$ is birational and finite (as a composition of a closed immersion and a finite morphism). By [GW10, Corollary 12.88], since X is normal, $X' \to X$ is an isomorphism. Hence σ is a section of π, so T/X is trivial. □

Lemma 3.10. *Let X be a proper variety over a finite field and Y/X be a finite flat scheme. Let Z/X be proper. Then $Y(Z)$ is finite.*

Proof. Since $\mathrm{Mor}_X(Z, Y) = \mathrm{Mor}_Z(Z, Y \times_X Z)$, one can assume $Z = X$. So we have to show that there are only finitely many sections to $\pi : Y \to X$. Such a section corresponds to an $\mathscr{O}_X$-algebra map $\pi_*\mathscr{O}_Y \to \mathscr{O}_X$. But $\mathrm{H}^0_{\mathrm{Zar}}(X, \mathscr{H}\mathrm{om}_X(\pi_*\mathscr{O}_Y, \mathscr{O}_X))$ is finite by the coherence theorem [EGAIII$_1$, Thm. (3.2.1)] as it is a finite dimensional vector space over a finite field. □

Lemma 3.11. *Let $Y \to X$ be an alteration of proper integral varieties with X normal, and G/X be a finite flat commutative group scheme. Then $\ker(\mathrm{H}^1_{\mathrm{fppf}}(X, G) \to \mathrm{H}^1_{\mathrm{fppf}}(Y, G))$ is finite. Hence $\mathrm{H}^1_{\mathrm{fppf}}(X, G)$ is finite if $\mathrm{H}^1_{\mathrm{fppf}}(Y, G)$ is.*

Proof. If $Y \to X$ is a blow-up, the kernel is trivial by Lemma 3.9 since a blow-up is birational. Hence the statement holds for blow-ups.

By [RG71, Théorème 5.2.2], there is a blow-up $f : X' \to X$ such that $Y' := Y \times_X X'$ is flat over X'. Since a normalization morphism of integral schemes is birational [Liu06, Proposition 4.1.22], one can assume X' normal. There is a commutative diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & \ker(\mathrm{H}^1_{\mathrm{fppf}}(X,G) \to \mathrm{H}^1_{\mathrm{fppf}}(Y,G)) & \longrightarrow & \mathrm{H}^1_{\mathrm{fppf}}(X,G) \\ & & \downarrow & & \downarrow f^* \\ 0 & \longrightarrow & \ker(\mathrm{H}^1_{\mathrm{fppf}}(X',G) \to \mathrm{H}^1_{\mathrm{fppf}}(Y',G)) & \longrightarrow & \mathrm{H}^1_{\mathrm{fppf}}(X',G) \end{array}$$

By the snake lemma, since $\ker f^*$ is finite as f is a blow-up, $\ker(\mathrm{H}^1_{\mathrm{fppf}}(X,G) \to \mathrm{H}^1_{\mathrm{fppf}}(Y,G))$ is finite if we can show that $\ker(\mathrm{H}^1_{\mathrm{fppf}}(X',G) \to \mathrm{H}^1_{\mathrm{fppf}}(Y',G))$ is finite. Hence, we can assume $Y \to X$ finite flat.

Let $T \to X$ be in the kernel, i. e., it is a G-torsor on X trivial when pulled back to Y. Choose a section $\sigma : Y \to T \times_X Y$; there are only finitely many of them by Lemma 3.10. Two such sections differ by an element of $G(Y)$. Since the base change $T \times_X (Y \times_X Y) \to Y \times_X Y$ is a G-torsor, one can take the 1-cocycle

$$\tau := d^0(\sigma) = \mathrm{pr}_0^*(\sigma) - \mathrm{pr}_1^*(\sigma) \in G(Y \times_X Y).$$

The section τ corresponds to the isomorphism class of the G-torsor T by the descent theory for the fppf covering $\{Y \to X\}$: As $\mathrm{H}^1_{\mathrm{fppf}}(-,G)$ can be computed by Čech cohomology and as the class of T in $\mathrm{H}^1_{\mathrm{fppf}}(X,G) = \check{H}^1_{\mathrm{fppf}}(X,G) = \varinjlim_{\mathscr{U}} \check{H}^1_{\mathrm{fppf}}(\mathscr{U},G)$ (the colimit taken over the coverings of X; the natural morphism from the first Čech cohomology to the first derived functor cohomology is always an isomorphism) is trivialized by the covering $\{Y \to X\}$, it can be represented as the 1-cocycle $\tau = d^0(\sigma)$, which is a 1-coboundary:

$$\check{H}^1(\{Y \to X\}, G) = \frac{\ker\left(G(Y \times_X Y) \xrightarrow{d^1} G(Y \times_X Y \times_X Y)\right)}{\mathrm{im}\left(G(Y) \overset{d^0}{\hookrightarrow} G(Y \times_X Y)\right)}$$

But by Lemma 3.10, $G(Y \times_X Y)$ is finite. □

Lemma 3.12. *Let X be an integral scheme with function field K and G/X be a finite flat group scheme. Let $H_K \hookrightarrow G_K$ be a finite flat group scheme. Then there is a blow-up $\tilde{X}/X$ such that H_K extends to a finite flat subgroup scheme of $G \times_X \tilde{X}$.*

Proof. Let $H \hookrightarrow G$ be the schematic closure of $H_K \hookrightarrow G$. The morphism $H \to G \to X$ is finite as a composition of a closed immersion and a finite morphism. By [RG71, Théorème 5.2.2], there is a blow-up $X' \to X$ such that $H' := H \times_X X' \to X'$ is flat. Then, H' is the schematic closure of $H_K \hookrightarrow G' := G \times_X X'$. So one can assume H/X finite flat.

Let $Y \to X$ be finite flat. Since the morphism is affine, locally, one has the diagram

$$\begin{array}{ccc} A & \hookrightarrow & A \otimes_R \mathrm{Quot}(R) \\ \uparrow & & \uparrow \\ R & \hookrightarrow & \mathrm{Quot}(R). \end{array}$$

Here, the upper horizontal arrow is injective by flatness of $R \to A$. Hence Y is the schematic closure of Y_K in Y.

By flatness, the schematic closure of $H_K \times_K H_K$ in $G \times_X G$ is $H \times_X H$. By the universal property of the schematic closure [GW10, (10.8)], one has the factorization

$$\begin{array}{ccc} H_K \times_K H_K & \xrightarrow{\mu} & H_K \\ \downarrow & & \downarrow \\ H \times_X H & \overset{\mu}{\dashrightarrow} & H \\ \downarrow & & \downarrow \\ G \times_X G & \xrightarrow{\mu} & G, \end{array}$$

for the multiplication μ, and similar for the inverse and unit section. □

Lemma 3.13. *Let X be a proper integral variety over a field and G/X be a finite flat commutative group scheme. After an alteration $X' \to X$, there exists a filtration of G by finite flat group schemes with subquotients of prime order.*

Proof. Over the algebraic closure of the function field of X, there is such an filtration since the only simple objects in the category of finite flat group schemes of p-power order are μ_p, $\mathbf{Z}/p$ and α_p. Since everything is of finite presentation, these are defined over a finite extension of the function field [GW10, Corollary 10.79]. Now take the normalization in this finite extension of function fields and use Lemma 3.12. □

Theorem 3.14. *Let X be a proper integral normal variety over a finite field and G/X be a finite flat commutative group scheme. Then $\mathrm{H}^1_{\mathrm{fppf}}(X, G)$ is finite.*

Proof. By Lemma 3.13, Lemma 3.11 and the long exact cohomology sequence one can assume G of prime order p (since the case of G/X étale is easily dealt with). Since then G is simple by [Sha86, p. 38] and since $F \circ V = [p] = 0$ by [Sha86, p. 62] and [Mum70, p. 141], either $V = 0$ or $F = 0$ on G.

If $V = 0$, by [de 93, Proposition 2.2], there is a short exact sequence

$$0 \to G \to \mathscr{L} \to \mathscr{M} \to 0$$

with vector bundles $\mathscr{L}, \mathscr{M}$. By the coherence theorem [EGAIII$_1$, Thm. (3.2.1)], as X is proper and lives over a finite ground field, and by comparison of Zariski and fppf cohomology [Mil80, Proposition III.3.7], the long exact cohomology sequence shows that $\mathrm{H}^i_{\mathrm{fppf}}(X, G)$ is finite.

If $F = 0$, after replacing X by an alteration by Lemma 3.11 as in the proof of Lemma 3.13, one can assume that G is isomorphic to μ_p over the generic point. Since for $Y, Z/X$ of finite presentation such that $Y_K \cong Z_K$, there is a non-empty open subscheme $U \hookrightarrow X$ such that $Y_U \cong Z_U$, there is a non-empty open subscheme $U \hookrightarrow X$ such that $G_U \cong \mu_{p,U}$. By [de 96], there is an alteration $f : X' \to X$ such that X' is regular. By Corollary 3.8, $\mathrm{H}^1_{\mathrm{fppf}}(f^{-1}(U), \mu_p)$ is finite. By Lemma 3.9, $\mathrm{H}^1_{\mathrm{fppf}}(X', G \times_X X')$ is finite, so by Lemma 3.11, $\mathrm{H}^1_{\mathrm{fppf}}(X, G)$ is finite. □

4 Isogeny invariance of finiteness of Ш, the p-part

In this section, we extend [Kel16, p. 240, Theorem 4.31] to p^∞-torsion.

Theorem 4.1. *Let X/k be a proper variety over a finite field k and $f : \mathscr{A} \to \mathscr{A}'$ be an isogeny of Abelian schemes over X. Let p be an arbitrary prime. Assume f étale if $p \neq \operatorname{char} k$. Then $Ш(\mathscr{A}/X)[p^\infty]$ is finite if and only if $Ш(\mathscr{A}'/X)[p^\infty]$ is finite.*

Proof. In the case where ℓ is invertible on X and f is étale (i. e., of degree invertible on X), this is [Kel16, p. 240, Theorem 4.31].

Now assume $p = \operatorname{char} k$. The short exact sequence of flat sheaves Lemma 3.4 yields an exact sequence in cohomology

$$\mathrm{H}^1_{\mathrm{fppf}}(X, \ker(f)) \to \mathrm{H}^1_{\mathrm{fppf}}(X, \mathscr{A}) \xrightarrow{f} \mathrm{H}^1_{\mathrm{fppf}}(X, \mathscr{A}')$$

and note that $\mathrm{H}^1_{\mathrm{fppf}}(X, \mathscr{A}) = \mathrm{H}^1_{\text{ét}}(X, \mathscr{A}) = Ш(\mathscr{A}/X)$ by Lemma 3.5 since $\mathscr{A}/X$ is smooth, and that $\mathrm{H}^1_{\mathrm{fppf}}(X, \ker(f))$ is finite by Theorem 3.14. Note that all groups are torsion (the Tate-Shafarevich groups by [Kel16, p. 224, Proposition 4.1]), hence the sequence stays exact after taking p^∞-torsion. So $Ш(\mathscr{A}/X)[p^\infty]$ is finite if $Ш(\mathscr{A}'/X)[p^\infty]$ is.

For the converse, note that by [Kel19, Proposition 2.19], there is a polarization $\lambda : \mathscr{A}^t \to \mathscr{A}$. Hence, the argument above for λ and λ^t implies that $Ш(\mathscr{A}^t/X)[p^\infty]$ is finite iff $Ш(\mathscr{A}/X)[p^\infty]$ is, and analogously for $Ш(\mathscr{A}'/X)[p^\infty]$. Taking the dual Kummer sequence $0 \to \ker(f^t) \to \mathscr{A}'^t \to \mathscr{A}^t \to 0$ yields an exact sequence

$$\mathrm{H}^1_{\mathrm{fppf}}(X, \ker(f^t)) \to Ш(\mathscr{A}'^t/X) \to Ш(\mathscr{A}^t/X).$$

By the same argument as above, $Ш(\mathscr{A}'^t/X)[p^\infty]$ is finite if $Ш(\mathscr{A}^t/X)[p^\infty]$ is if $Ш(\mathscr{A}/X)[p^\infty]$ is. So $Ш(\mathscr{A}'/X)[p^\infty]$ is finite. □

5 Descent of finiteness of Ш, the p-part

In this section, we extend [Kel16, p. 238, Theorem 4.29] to p^∞-torsion.

Lemma 5.1. *Let $\mathscr{A}/X$ be an Abelian scheme over a proper variety X over a finite field of characteristic p. Then $Ш(\mathscr{A}/X)[p^\infty]$ is cofinitely generated.*

Recall that $Ш(\mathscr{A}/X)$ was defined as $\mathrm{H}^1_{\text{ét}}(X, \mathscr{A})$ in [Kel16, p. 225, Definition 4.2].

Proof. The long exact cohomology sequence associated to the Kummer sequence Lemma 3.4 gives us a surjection

$$\mathrm{H}^1_{\mathrm{fppf}}(X, \mathscr{A}[p^n]) \twoheadrightarrow \mathrm{H}^1_{\mathrm{fppf}}(X, \mathscr{A})[p^n] \to 0$$

Now, since $\mathscr{A}/X$ is a smooth group scheme, Lemma 3.5 gives us an isomorphism $\mathrm{H}^1_{\mathrm{fppf}}(X, \mathscr{A}) = \mathrm{H}^1_{\text{ét}}(X, \mathscr{A})$, which by definition equals $Ш(\mathscr{A}/X)$. By Theorem 3.14, $\mathrm{H}^1_{\mathrm{fppf}}(X, \mathscr{A}[p^n])$ is finite since $X/\mathbf{F}_q$ is proper. From this, one sees that $\mathrm{H}^1_{\text{ét}}(X, \mathscr{A})[p]$ is finite. Hence $Ш(\mathscr{A}/X)[p^\infty]$ is cofinitely generated by [Kel19, Lemma 2.38]. □

Lemma 5.2 (existence of trace morphism)**.** *Let $f : X' \to X$ be a finite étale morphism of constant degree d and let $\mathscr{F}$ be an fppf sheaf on X. Then there is a trace map $\mathrm{Tr}_f : f_*f^*\mathscr{F} \to \mathscr{F}$, functorial in $\mathscr{F}$, such that $\varphi \mapsto \mathrm{Tr}_f \circ f_*(\varphi)$ is an isomorphism $\mathrm{Hom}_{X'}(\mathscr{F}', f^*\mathscr{F}) \to \mathrm{Hom}_X(\pi_*\mathscr{F}', \mathscr{F})$ for any fppf sheaf $\mathscr{F}'$ on X'. Thus, $f_* = f_!$, that is, f_* is left adjoint to f^*, and Tr_f is the adjunction map. The composites*

$$\mathscr{F} \to f_*f^*\mathscr{F} \overset{\mathrm{Tr}_f}{\to} \mathscr{F} \quad \textit{and} \quad \mathrm{H}^r_{\mathrm{fppf}}(X, \mathscr{F}) \xrightarrow{f^*} \mathrm{H}^r_{\mathrm{fppf}}(X', f^*\mathscr{F}) \xrightarrow{\mathrm{can}} \mathrm{H}^r_{\mathrm{fppf}}(X, f_*f^*\mathscr{F}) \xrightarrow{\mathrm{Tr}_f} \mathrm{H}^r_{\mathrm{fppf}}(X, \mathscr{F})$$

are multiplication by d.

Proof. On may copy the proof of [Mil80, p. 168, Lemma V.1.12] almost verbatim: Let $\mathscr{F}$ be a fppf sheaf on X. Let $X'' \to X$ be finite Galois with Galois group G factoring as $X'' \to X' \to X$; $X'' \to X'$ is Galois with Galois group $H \leq G$. For any U/X flat, we have $\Gamma(U, \mathscr{F}) \hookrightarrow \Gamma(U', \mathscr{F}) \hookrightarrow \Gamma(U'', \mathscr{F})$ and $\Gamma(U, \mathscr{F}) \xrightarrow{\sim} \Gamma(U'', \mathscr{F})^G$, where $U' = U \times_X X'$ and $U'' = U \times_X X''$. For a section $s \in \Gamma(U, f_*f^*\mathscr{F}) := \Gamma(U', \mathscr{F})$, we define

$$\mathrm{Tr}_f(s) := \sum_{\sigma \in G/H} \sigma(s|_{U''});$$

as this is fixed by G, it may be regarded as an element of $\Gamma(U, \mathscr{F}) \xrightarrow{\sim} \Gamma(U'', \mathscr{F})^G$. Clearly, Tr_f defines a morphism $f_*f^*\mathscr{F} \to \mathscr{F}$ such that its composite with $\mathscr{F} \to f_*f^*\mathscr{F}$ is multiplication by the degree d of f.

If X' is a disjoint union of d copies of X, obviously $\mathrm{Hom}_{X'}(\mathscr{F}', f^*\mathscr{F}) \to \mathrm{Hom}_X(f_*\mathscr{F}', \mathscr{F})$, and one may reduce the question to this split case by passing to a finite étale covering of X, for example to $X'' \to X$, and using the fact that Hom is a sheaf.

In

$$\mathrm{H}^r_{\mathrm{fppf}}(X, \mathscr{F}) \xrightarrow{f^*} \mathrm{H}^r_{\mathrm{fppf}}(X', f^*\mathscr{F}) \xrightarrow{\mathrm{can}} \mathrm{H}^r_{\mathrm{fppf}}(X, f_*f^*\mathscr{F}) \xrightarrow{\mathrm{Tr}_f} \mathrm{H}^r_{\mathrm{fppf}}(X, \mathscr{F})$$

the composite of the first two maps is induced by $\mathscr{F} \to f_*f^*\mathscr{F}$, and the composite of all three is induced by $(\mathscr{F} \to f_*f^*\mathscr{F} \xrightarrow{\mathrm{Tr}_f} \mathscr{F}) =$ multiplication by d. □

Theorem 5.3. *Let p be a prime and X be a scheme of characteristic p. Let $f : X' \to X$ be a proper, surjective, generically étale morphism of generical degree prime to p of regular, integral, separated varieties over a finite field. Let $\mathscr{A}$ be an abelian scheme on X and $\mathscr{A}' := f^*\mathscr{A} = \mathscr{A} \times_X X'$. If $Ш(\mathscr{A}'/X')[p^\infty]$ is finite, so is $Ш(\mathscr{A}/X)[p^\infty]$.*

Proof. The same proof as in [Kel16, Theorem 4.29] works, one only needs $Ш(\mathscr{A}/X)[p^\infty]$ to be cofinitely generated in Step 2, which is Lemma 5.1. The trace morphism in Step 3 for fppf cohomology comes from Lemma 5.2. Note that the proof given there does not need the regularity of X, X' and that varieties over a field are excellent by [Liu06, Corollary 2.40 (a)]. For the convenience of the reader, we reproduce the proof of [Kel16, Theorem 4.29] adapted to our situation here:

Step 1: $\mathbf{H}^1_{\mathbf{fppf}}(X, f_*\mathscr{A}')[p^\infty]$ is finite. This follows from the low terms exact sequence

$$0 \to \mathrm{H}^1_{\mathrm{fppf}}(X, f_*\mathscr{A}') \to \mathrm{H}^1_{\mathrm{fppf}}(X', \mathscr{A}')$$

associated to the Leray spectral sequence $\mathrm{H}^p_{\mathrm{fppf}}(X, \mathrm{R}^q f_* \mathscr{A}') \Rightarrow \mathrm{H}^{p+q}_{\mathrm{fppf}}(X', \mathscr{A}')$ and the finiteness of

$$\mathrm{H}^1_{\mathrm{fppf}}(X', \mathscr{A}')[p^\infty] = Ш(\mathscr{A}'/X')[p^\infty].$$

Step 2: The theorem holds if there is a trace morphism. Since by Lemma 5.2 there is a trace morphism $f_* f^* \mathscr{A} \to \mathscr{A}$ such that the composition with the adjunction morphism

$$\mathscr{A} \to f_* f^* \mathscr{A} \to \mathscr{A}$$

is multiplication by $\deg f \neq 0$, the finiteness of $\mathrm{H}^1_{\mathrm{fppf}}(X, \mathscr{A})[p^\infty]$ follows from that of $\mathrm{H}^1_{\mathrm{fppf}}(X, f_*\mathscr{A}')[p^\infty]$ because both groups are cofinitely generated by Lemma 5.1.

Step 3: Proof of the theorem in the general case. Let η be the generic point of X. Define X'_η by the commutativity of the cartesian diagram

$$\begin{array}{ccc} X'_\eta & \overset{g'}{\hookrightarrow} & X' \\ \downarrow f_\eta & & \downarrow f \\ \{\eta\} & \overset{g}{\hookrightarrow} & X. \end{array} \tag{5.1}$$

Since f is generically étale, we can apply Lemma 5.2 to f_η in this commutative diagram. From the commutativity of that diagram, the kernel of $f^* : \mathrm{H}^1_{\mathrm{fppf}}(X, \mathscr{A}) \to \mathrm{H}^1_{\mathrm{fppf}}(X', \mathscr{A}')$ is contained in the kernel of the composition

$$\mathrm{H}^1_{\mathrm{fppf}}(X, \mathscr{A}) \overset{g^*}{\to} \mathrm{H}^1_{\mathrm{fppf}}(\{\eta\}, \mathscr{A}_\eta) \overset{f^*_\eta}{\to} \mathrm{H}^1_{\mathrm{fppf}}(X'_\eta, \mathscr{A}'_{X'_\eta}),$$

so it suffices to show that the first arrow g^* is injective. But by the Néron mapping property $\mathscr{A} \overset{\sim}{\longrightarrow} g_* g^* \mathscr{A}$ [Kel16, Theorem 3.3] (for the *étale* topology!), $\mathrm{H}^1_{\text{ét}}(X, \mathscr{A}) \overset{\sim}{\longrightarrow} \mathrm{H}^1_{\text{ét}}(X, g_* \mathscr{A}_\eta)$. However, the Leray spectral sequence $\mathrm{H}^p_{\text{ét}}(X, \mathrm{R}^q g_* \mathscr{A}_\eta) \Rightarrow \mathrm{H}^{p+q}_{\text{ét}}(\{\eta\}, \mathscr{A}_\eta)$ gives an injection

$$0 \to \mathrm{H}^1_{\text{ét}}(X, g_* \mathscr{A}_\eta) \to \mathrm{H}^1_{\text{ét}}(\{\eta\}, \mathscr{A}_\eta).$$

But because $\mathscr{A}/X$ and $\mathscr{A}_\eta/\{\eta\}$ are smooth commutative group schemes, their étale cohomology agrees with their flat cohomology, see Lemma 3.5, and the comparison of topology morphisms are functorial. □

Theorem 5.4 (Stein factorization, alteration = finite ∘ modification)**.** *Let $f : X' \to X$ be a proper morphism of Noetherian schemes. Then one can factor f into $g \circ f'$, where $f' : X' \to Y := \mathbf{Spec}_X f_* \mathscr{O}_{X'}$ is a proper morphism with connected fibers, and $g : Y \to X$ is a finite morphism. If f is an alteration, f' is birational and proper (a* modification*).*

Proof. See [EGAIII$_1$, Thm. 4.3.1] for the statement on the existence of the factorization, which includes $Y = \mathbf{Spec}_X f_* \mathscr{O}_{X'}$.

Assume now that f is an alteration. If $U \subseteq X$ is an open subscheme such that $f|_U$ is finite (in particular affine), one may shrink U such that it is affine, so by finiteness of g, $g : g^{-1}(U) \to U$ is finite and can be written as $\operatorname{Spec} B \to \operatorname{Spec} A$. From the statement of Stein factorization, $g^{-1}(U) = \mathbf{Spec}_U f_* \mathscr{O}_{f^{-1}(U)}$, but f' has geometrically connected fibers, so $f'_* \mathscr{O}_{X'} = \mathscr{O}_Y$, so $f'|_{g^{-1}(U)}$ is an isomorphism because it is affine. □

We also remove the hypotheses that f is generically étale and has degree prime to ℓ if ℓ is invertible on the base scheme in [Kel16, Theorem 4.29]:

Theorem 5.5. *Let $f : X' \to X$ be a proper, surjective, generically finite morphism of regular, integral, separated varieties over a finite field. Let $\mathscr{A}$ be an abelian scheme on X and $\mathscr{A}' := f^* \mathscr{A} = \mathscr{A} \times_X X'$. Let ℓ be invertible on X. If $Ш(\mathscr{A}'/X')[\ell^\infty]$ is finite, so is $Ш(\mathscr{A}/X)[\ell^\infty]$.*

Proof. By the Stein factorization Theorem 5.4, f factors as a proper, surjective, birational morphism followed by a finite morphism. In particular, it is a generically étale alteration. The finite morphism factors as a finite purely inseparable morphism followed by a finite generically étale morphism. We prove the finiteness assertion of the theorem for all such morphisms separately:

If f is generically étale, this is Theorem 5.3. If f is a proper, surjective, birational morphism, it is generically an isomorphism, i.e., generically étale of degree 1.

If f is a universal homeomorphism, the étale sites of X and X' are equivalent by f^* and f_* by [SGA4.2, VIII.1.1]. In particular, the étale cohomology groups $Ш(\mathscr{A}/X) = \mathrm{H}^1_{\text{ét}}(X, \mathscr{A})$ and $Ш(\mathscr{A}'/X') = \mathrm{H}^1_{\text{ét}}(X', f^*\mathscr{A})$ are isomorphic via f^*. □

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