

Transfer of quadratic forms and their Stiefel-Whitney invariants

Timo Keller

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Abstract

Let L/K be a finite extension of fields of characteristic $\neq 2$. We prove that the $t = 2^{n-1}$ -th Stiefel-Whitney invariant of an arbitrary Scharlau transfer of an element of the n -th power of the fundamental ideal of the Grothendieck-Witt ring of L is the corestriction of the t -th Stiefel-Whitney invariant of that element. Together with our second main result, the computation of the trace forms of field extensions obtained by adjoining a root of $X^{km} + a_k X^k + a_0$, this gives a formula for the absolute Stiefel-Whitney invariant of the transfer of quadratic forms for these field extensions.

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1 Introduction

Let L/K be a finite separable field extension and q an element of the Grothendieck-Witt ring $\widehat{W}(L)$ of L , which is the Grothendieck ring of the commutative semiring of quadratic forms over L with addition the orthogonal direct sum and multiplication the tensor product. On $\widehat{W}(L)$, one can define an invariant w , called the absolute Stiefel-Whitney invariant, with values in the Milnor K -theory of L modulo 2. This is in analogy with the Stiefel-Whitney classes of vector bundles on a topological space or the Chern character of vector bundles on schemes, where the famous Grothendieck-Riemann-Roch theorem describes the Chern classes of the push-forward of a vector bundle in terms of the push-forward of the Chern classes of the vector bundle with the Todd class of the canonical bundle appearing in the formula. Analogous to the push-forward of vector bundles is the Scharlau transfer of quadratic forms: To a surjective K -linear map $s : L \rightarrow K$ and a quadratic form q over L , its transfer $s(q)$ is the composition $s \circ q : L^n \rightarrow L \rightarrow K$. If L/K is separable, the trace form $\mathrm{Tr}_{L/K}(\langle 1 \rangle)$ is non-degenerate and a canonical Scharlau transfer.

Kahn [Kah84] found the Grothendieck-Riemann-Roch type relation

$$w(\mathrm{Tr}(q)) = \mathcal{N}(w(q)) \cdot w(\mathrm{Tr}(\langle 1 \rangle))^{\dim q}$$

between $w(\mathrm{Tr}(q))$ and $w(q)$. Here, \mathcal{N} is Evens' [Eve63] multiplicative norm $\mathcal{N} : \mathrm{H}^{**}(L) \rightarrow \mathrm{H}^{**}(K)$ on the Galois cohomology rings $\mathrm{H}^{**}(L) = \prod_{i=0}^{\infty} \mathrm{H}^i(L, \mathbf{Z}/2)$, $\dim q$ is the dimension of q , and $w(\mathrm{Tr}(\langle 1 \rangle))$ the absolute Stiefel-Whitney invariant of the trace form of L/K .

According to this formula, the calculation of $w(\mathrm{Tr}(q))$ is split up into two parts: On the one hand, one has to find an easy formula for the multiplicative norm \mathcal{N} , on the other hand, one has to calculate $w(\mathrm{Tr}(\langle 1 \rangle))$. For the first Stiefel-Whitney invariant, there were already results by Bourbaki. Serre [Ser84] considered the second Stiefel-Whitney invariant and obtained a formula in 1984. Since then, there had been no further results.

In this article, we address both questions for special infinite families:

Theorem 1 (Theorem 2.3). *Let $n \geq 1$, $t = 2^{n-1}$. Then one has, for all $q \in \hat{I}_L^n$ and all Scharlau transfers $s : L \rightarrow K$,*

$$w_k(s(q)) = \mathrm{cor}(w_k(q))$$

for all k not divisible by t and for $k = t$. Here, both sides are 0 if k is not divisible by t .

Theorem 2 (Theorem 3.9). *Let $L = K(a)/K$ be a finite separable field extension of degree $kn > 1$ with minimal polynomial $f(X) = X^{kn} + a_k X^k + a_0 \in K[X]$ of a over K . Let*

$$c(f) = (1 - n)^{n-1} a_k^n + n^n a_0^{n-1}.$$

Let $n > 2, k > 1$ and $a_k \neq 0$. Let $\text{char } K \nmid kn(n-1)$. Then one has $\text{Tr}_{L/K}(\langle 1 \rangle) =$

$$\begin{cases} \langle kn, k(1-n)a_k, -a_0 kc(f), -n(1-n)a_k kc(f) \rangle + \left(\frac{k}{2}n - 2\right) \langle 1, -1 \rangle, & k \text{ even, } n \text{ odd} \\ \langle kn, k(1-n)a_k, -ka_0 a_k, -kn(1-n) \rangle + \left(\frac{k}{2}n - 2\right) \langle 1, -1 \rangle, & k \text{ even, } n \text{ even} \\ \langle kn, -c(f)kn \rangle + \frac{kn-2}{2} \langle 1, -1 \rangle, & k \text{ odd, } n \text{ even} \\ \langle kn, k(1-n)a_k, -c(f)kn(1-n)a_k \rangle + \left(\frac{kn-3}{2}\right) \langle 1, -1 \rangle, & k \text{ odd, } n \text{ odd.} \end{cases}$$

(The formulas for the cases $n = 2, k = 1, n > 2, k = 1$, and $n = 2, k > 1$ can be found below.)

Notation. Denote the cokernel of $A \xrightarrow{n} A$ by A/n and its kernel by $A[n]$. Denote Milnor K -theory of a field K by $K_i^M(K)$ and $K_*^M(K) = \bigoplus_{i \geq 0} K_i^M(K)$ and $K_{**}^M(K) = \prod_{i \geq 0} K_i^M(K)$. Quadratic forms are denoted by $\langle a_1, \dots, a_n \rangle$, and for $a_i \in K^\times$ the Pfister form $\langle\langle a_1, \dots, a_n \rangle\rangle = \prod_{i=1}^n \langle 1, -a_i \rangle$. For a field K , let $H^i(K) = H^i(K, \mathbf{Z}/2)$ be Galois cohomology, $H^*(K) = \bigoplus_{i \geq 0} H^i(K)$ and $H^{**}(K) = \prod_{i \geq 0} H^i(K)$. Canonical isomorphisms are often denoted by “ $=$ ”. Denote the Grothendieck-Witt ring of a field K by $\widehat{W}(K)$ with fundamental ideal \hat{I}_K and its Witt ring by $W(K)$. For a finite separable field extension L/K denote the transfer $W(L) \rightarrow W(K)$ resp. $\widehat{W}(L) \rightarrow \widehat{W}(K)$ by $\text{Tr}_{L/K}$. Denote the absolute Stiefel-Whitney invariant by $w : \widehat{W}(K) \rightarrow K_{**}^M(K)/2 = H^{**}(K)$ (the equality by the Milnor conjecture) and its graded components by w_i .

2 The multiplicative norm and Stiefel-Whitney invariants

All fields (except our coefficient ring $\mathbf{Z}/2$) are assumed to have characteristic $\neq 2$. First, we partially calculate Evens’ multiplicative norm [Eve63] generalising *loc. cit.*, p. 63, Theorem 1. Let us recall briefly the construction of the multiplicative norm $\mathcal{N} : H^{**}(H) \rightarrow H^{**}(G)$ for $H \leq G$ an open subgroup of finite index n of a profinite group G in the case of $\mathbf{Z}/2$ -coefficients, which carry the trivial G -action.

Let τ_1, \dots, τ_n be a system of left coset representatives of G/H . The choice of the $(\tau_i)_{i=1}^n$ induces an injection $\Phi : G \hookrightarrow S_n \wr H$ in the wreath product $S_n \wr H = H^n \rtimes S_n$, called the monomial representation.

In the following, all tensor products are taken over $\mathbf{Z}/2$. If C_\bullet is a complex of $\mathbf{Z}/2[H]$ -modules, $(C_\bullet)^{\otimes n}$ becomes a complex of $\mathbf{Z}/2[S_n \wr H]$ -modules via

$$\begin{aligned} (\sigma_1 \times \sigma_2 \times \dots \times \sigma_n)(a_1 \otimes a_2 \otimes \dots \otimes a_n) &= \sigma_1 a_1 \otimes \sigma_2 a_2 \otimes \dots \otimes \sigma_n a_n, \quad \sigma_i \in H \\ \pi(a_1 \otimes a_2 \otimes \dots \otimes a_n) &= a_{\pi^{-1}(1)} \otimes a_{\pi^{-1}(2)} \otimes \dots \otimes a_{\pi^{-1}(n)}, \quad \pi \in S_n. \end{aligned}$$

Pulling back with the monomorphism Φ , $(C_\bullet)^{\otimes n}$ becomes a complex of $\mathbf{Z}/2[G]$ -modules. Let $P_\bullet \xrightarrow{\varepsilon} \mathbf{Z}/2 \rightarrow 0$ and $R_\bullet \xrightarrow{\eta} \mathbf{Z}/2 \rightarrow 0$ be a $\mathbf{Z}/2[H]$ -projective, respectively a $\mathbf{Z}/2[S_n \wr H]$ -projective resolution of $\mathbf{Z}/2$. Then

$$R_\bullet \otimes (P_\bullet)^{\otimes n} \xrightarrow{\eta \otimes \varepsilon^{\otimes n}} \mathbf{Z}/2 \rightarrow 0$$

is a $\mathbf{Z}/2[S_n \wr H]$ -projective resolution of $\mathbf{Z}/2$. Let $f = (f_i)$, $f_i : P_i \rightarrow \mathbf{Z}/2$ be a cocycle representing a cohomology class $\alpha = (\alpha_i) \in H^{**}(H)$. Then the **multiplicative norm** $\mathcal{N}(\alpha) \in H^{**}(G)$ is defined as the cohomology class represented by $\Phi^*(1 \wr f^{\otimes n})$, where

$$1 \wr f^{\otimes n} := \eta \otimes f^{\otimes n} \in \text{Hom}_{\mathbf{Z}/2[S_n \wr H]}(R_\bullet \otimes (P_\bullet)^{\otimes n}, \mathbf{Z}/2).$$

One checks that this is well-defined. Note that since our coefficient module $\mathbf{Z}/2$ is a commutative ring of characteristic 2, we do not have to worry about signs.

Theorem 2.1. *Let G be a profinite group with open subgroup H of index $n = [G : H]$. Assume $x \in H^{**}(H)$ can be written as $x = 1 + x_t + \sum_{j>t} x_j$, with $t > 0$ and $x_j \in H^j(H)$. Then one has*

$$\mathcal{N}(x) = 1 + \text{cor}_G^H(x_t) + \sum_{k>t} \mathcal{N}_k(x) \quad \text{with } \mathcal{N}_k(x) \in H^k(G),$$

where in the sum only such k can occur which can be written as a sum with at most n summands of j such that $x_j \neq 0$.

Proof. First assume G finite. Let $P_\bullet \xrightarrow{\varepsilon} \mathbf{Z}/2 \rightarrow 0$ and $R_\bullet \xrightarrow{\eta} \mathbf{Z}/2 \rightarrow 0$ be as above.

Choose a cocycle $f = (f_i)$, $f_i : P_i \rightarrow \mathbf{Z}/2$ representing x . One can assume $f_i = 0$ for those i such that $x_i = 0$. Then, obviously, $1 \wr f^{\otimes n} = \eta \otimes f^{\otimes n}$ is trivial in degrees which cannot be written as a sum with (at most) n summands of j such that $x_j \neq 0$. (It does not matter if we write “at most” or not since $x_0 = 1 \neq 0$.)

One has $\mathcal{N}_i(x) = \mathcal{N}_i(1 + x_i)$, and this is equal to $\text{cor}_G^H(x_i)$ by [CTVE⁺03], p. 118 f., Theorem 6.3.5 6.

Reduce the general case to this using projective limits. \square

Next, we partially calculate the Stiefel-Whitney invariants of quadratic forms from the n -th power of the fundamental ideal $\hat{I}_K \subset \widehat{W}(K)$:

Proposition 2.2. *Let K be a field of characteristic $\neq 2$, $n \geq 1$, $t = 2^{n-1}$ and $q \in \hat{I}_K^n$. Then $w_k(q) \neq 0$ can only occur if $t \mid k$. Further, $w_t(\langle \alpha \rangle \cdot q) = w_t(q)$ for all $\alpha \in K^\times$.*

Proof. According to [Lam05], p. 28, Proposition 1.2, \hat{I}_K is additively generated by forms $\langle a \rangle - \langle 1 \rangle$, $a \in K^\times$. Therefore, every $q \in \hat{I}_K^n$ is a sum of forms of type $\langle a \rangle q$, where $q = \prod_{i=1}^n (\langle a_i \rangle - \langle 1 \rangle)$. For the Stiefel-Whitney invariants of such a form one has

$$\begin{aligned} w \left(\langle a \rangle \prod_{i=1}^n (\langle a_i \rangle - \langle 1 \rangle) \right) &= w \left((\langle a \rangle - \langle 1 \rangle) \prod_{i=1}^n (\langle a_i \rangle - \langle 1 \rangle) + \prod_{i=1}^n (\langle a_i \rangle - \langle 1 \rangle) \right) \\ &= (1 + \underbrace{\{a, a_1, \dots, a_n\} \cdot \{-1\}^{2t-n-1}}_{\text{in degree } 2t})^\pm (1 + \underbrace{\{a_1, \dots, a_n\} \cdot \{-1\}^{t-n}}_{\text{in degree } t})^\mp \end{aligned}$$

according to [Mil70], p. 329, Lemma 3.2 and the Whitney summation formula. Because of the formula for the geometric series, terms $\neq 0$ can occur only in degrees divisible by t . One has

$$w \left(\langle a \rangle \prod_{i=1}^n (\langle a_i \rangle - \langle 1 \rangle) \right) = 1 + \underbrace{\{a_1, \dots, a_n\} \cdot \{-1\}^{t-n}}_{\text{in degree } t} + \dots,$$

where \dots denotes the omission of terms in degrees $> t$. Hence

$$\begin{aligned} w_t \left(\langle a \rangle \prod_{i=1}^n (\langle a_i \rangle - \langle 1 \rangle) \right) &= \{a_1, \dots, a_n\} \cdot \{-1\}^{t-n} \\ &= w_t \left(\prod_{i=1}^n (\langle a_i \rangle - \langle 1 \rangle) \right) \quad \text{by } \textit{loc. cit.}, \text{ p. 329, Lemma 3.2.} \end{aligned}$$

Therefore the claim is proven for forms of the type $\langle a \rangle \prod_{i=1}^n (\langle a_i \rangle - \langle 1 \rangle)$. In the general case, the claim follows for sums $q_1 + \dots + q_r$ of such forms by the Whitney summation formula: One has

$$w_i(q_1 + \dots + q_r) = \sum_{\substack{i_1 + \dots + i_r = i \\ 0 \leq i_j}} w_{i_1}(q_1) \cdots w_{i_r}(q_r),$$

and if $t \nmid i$, there is for $i_1 + \dots + i_r = i$ a j with $t \nmid i_j$, so $w_{i_j}(q_j) = 0$. For $i = t$ one has

$$\begin{aligned} w_t(q_1 + \dots + q_r) &= \sum_{\substack{i_1 + \dots + i_r = t \\ 0 \leq i_j}} w_{i_1}(q_1) \cdots w_{i_r}(q_r) \\ &= \sum_{j=1}^r w_t(q_j), \end{aligned}$$

since for $i_1 + \dots + i_r = t$, $0 \leq i_j$ and $t \nmid i_j$ there is exactly one $i_j = t$ and the others equal 0. Hence

$$\begin{aligned} w_t(\langle \alpha \rangle \cdot (q_1 + \dots + q_r)) &= w_t(\langle \alpha \rangle \cdot q_1 + \dots + \langle \alpha \rangle \cdot q_r) \\ &= \sum_{j=1}^r w_t(\langle \alpha \rangle \cdot q_j) \\ &= \sum_{j=1}^r w_t(q_j) \\ &= w_t(\langle \alpha \rangle \cdot (q_1 + \dots + q_r)) \end{aligned}$$

since the Stiefel-Whitney invariants of $\langle \alpha \rangle \cdot q_j$ vanish in degrees not divisible by t if q_j is of the form $\langle \alpha_j \rangle \prod_{i=1}^n (\langle a_{ij} \rangle - \langle 1 \rangle)$, as we have seen above. \square

Combining the above theorem with Kahn's formula [Kah84], p. 224, Théorème 2

$$w(\mathrm{Tr}(q)) = \mathcal{N}(w(q)) \cdot w(\mathrm{Tr}(\langle 1 \rangle))^{\dim q}$$

yields

Theorem 2.3. *Let $n \geq 1$, $t = 2^{n-1}$. Then one has, for all $q \in \hat{I}_L^n$ and all Scharlau transfers $s : L \rightarrow K$,*

$$w_k(s(q)) = \mathrm{cor}(w_k(q))$$

for all k not divisible by t and for $k = t$. Here, both sides are 0 if k is not divisible by t .

Proof. Let $q \in \hat{I}_L^n$. There is a unique $\alpha \in L^\times$ such that $s(x) = \mathrm{Tr}_{L/K}(\alpha \cdot x)$ for all $x \in L$. One has

$$\begin{aligned} w(s(q)) &= w_k(\mathrm{Tr}_{L/K}(\langle \alpha \rangle \cdot q)) \\ &= \mathcal{N}(w(\langle \alpha \rangle \cdot q)) \cdot w(\mathrm{Tr}_{L/K}(\langle 1 \rangle))^{\dim \langle \alpha \rangle \cdot q} \\ &= \mathcal{N}(w(\langle \alpha \rangle \cdot q)) \\ &= \mathcal{N}\left(1 + \sum_{i=1}^{\infty} w_{ti}(q)\right) \quad \text{by Proposition 2.2.} \end{aligned}$$

Applying Theorem 2.1 yields the result. \square

Remark 2.4. Let L/K be a quadratic extension. If one writes for $1 + x \in \mathrm{H}^{**}(L)$ the multiplicative norm as in [Kah84], p. 230 f. as

$$\mathcal{N}(1 + x) = 1 + \mathcal{N}_1(x) + \mathcal{N}_2(x)$$

(for quadratic extensions there only occur \mathcal{N}_i with $i \leq 2$ according to [Kah84], p. 230), where \mathcal{N}_1 is the corestriction, there is a method for calculating $\mathcal{N}_2(x)$ in low degrees $< 2n_2$:

Write $1 + x = 1 + \sum_{i \geq 1} x_i$ with $x_i \in \mathrm{H}^{n_1}(L)$, $1 = n_1 < n_2 < \dots$. According to the addition formula [Kah84], p. 231, Proposition I.2.4. a) one has

$$\begin{aligned} \mathcal{N}_2(x) &= \mathcal{N}_2(x_1 + x_2 + \dots) \\ &= \mathcal{N}_2((1 + x_1) + (1 + x_2 + \dots)) \\ &= \mathcal{N}_2(1 + x_1) + \mathcal{N}_2(1 + x_2 + \dots) + \\ &\quad \mathrm{cor}((1 + x_1) \cdot (1 + x_2 + \dots)) + \mathrm{cor}((1 + x_1)) \cdot \mathrm{cor}(1 + x_2 + \dots), \end{aligned}$$

where the sum $\sum_{i > 2} x_i$ is denoted by \dots . Therefore, it suffices to calculate $\mathcal{N}_2(1 + x_1)$ and $\mathcal{N}_2(1 + x_2 + \dots)$.

By the addition formula, one has $\mathcal{N}_2(1 + y) = \mathcal{N}_2(1) + \mathcal{N}_2(y) + \mathrm{cor}(1) \cdot \mathrm{cor}(y) + \mathrm{cor}(1 \cdot y)$, and $\mathcal{N}_2(1)$ can be calculated since $0 = \mathcal{N}(1 + 1) = 1 + \mathrm{cor}(1) + \mathcal{N}_2(1)$. Therefore, one has to determine $\mathcal{N}_2(x_1)$ and $\mathcal{N}_2(x_2 + \dots)$.

But $\mathcal{N}(1 + x_1) = 1 + \mathrm{cor}(x_1) + \mathcal{N}_2(x_1)$ with $\mathcal{N}_i(x_1) \in \mathrm{H}^{i n_1}(K)$ according to [Kah84], p. 230, so it suffices to calculate $\mathcal{N}(1 + x_1)$. If $x_1 = \{a\} \in \mathbf{K}_1^{\mathbf{M}}(L)/2$, then

$$\mathcal{N}(1 + x_1) = w(\mathrm{Tr}_{L/K}(\langle a \rangle)) \cdot (w(\mathrm{Tr}_{L/K}(\langle 1 \rangle)))^{-1}$$

and the terms on the right hand side can for quadratic extensions be determined as follows: Write $L = K(\sqrt{c})/K$ and let $\alpha = a + b\sqrt{c} \in L^\times$. Then

$$w(\mathrm{Tr}_{L/K}(\langle \alpha \rangle)) \cong \begin{cases} 1 + \{-1\} + 0, & \text{if } a = 0 \\ 1 + \{c N_{L/K}(\alpha)\} + \{2a, -c N_{L/K}(\alpha)\}, & \text{if } a \neq 0. \end{cases}$$

Analogously, one can write $\mathcal{N}(1 + x_2 + \dots) = 1 + \mathrm{cor}(x_2 + \dots) + \mathcal{N}_2(x_2 + \dots)$ and calculate the left hand side and hence $\mathcal{N}_2(x_2 + \dots)$ using Theorem 2.1 in the relevant low degrees $< 2n_2$.

Note that Galois field extensions L/K of odd degree n pose no real problem, since for these the restriction $\mathrm{res} : \mathrm{H}^{**}(K) \rightarrow \mathrm{H}^{**}(L)$ is injective and by [CTVE⁺03], p. 118, Theorem 6.3.54 one has

$$\mathrm{res} \circ \mathcal{N}(x) = \prod_{\sigma \in \mathrm{Gal}(L/K)} \sigma x.$$

3 Trace forms

Finally, we calculate the trace forms of special field extensions. This is motivated by work of Serre [Ser84]. There are applications to embedding problems [Ser92], p. 99, Corollary 9.2.3 and inverse Galois theory [MM99], p. 312 ff.

Proposition 3.1. *Let L/K be a finite separable field extension with Galois hull \tilde{L}/K .*

(i) *If $[\tilde{L} : K]$ is odd, $\mathrm{Tr}_{L/K}(\langle 1 \rangle) = [L : K] \langle 1 \rangle$.*

(ii) *If L/K is Galois, one has $w(\mathrm{Tr}_{L/K}(\langle 1 \rangle)) \equiv 1 \pmod{\ker(\mathrm{res} : \mathbb{K}_{**}^M(K)/2 \rightarrow \mathbb{K}_{**}^M(L)/2)}$. If $[\tilde{L} : K]$ is odd, we have $w(\mathrm{Tr}_{L/K}(\langle 1 \rangle)) = 1$.*

Proof. (i): We first show the claim for $\tilde{L} = L$. By [Lam05], p. 210 f., Theorem 6.1, one has

$$r_{L/K}^*(\mathrm{Tr}_{L/K}(\langle 1 \rangle_L)) = \sum_{\sigma \in \mathrm{Gal}(L/K)} \langle 1 \rangle_L^\sigma = [L : K] \langle 1 \rangle_L = r_{L/K}^*([L : K] \langle 1 \rangle_K)$$

because of $\langle 1 \rangle_L^\sigma = \langle 1^{\sigma^{-1}} \rangle_L = \langle 1 \rangle_L$, see [Lam05], p. 210. So by Springer's theorem [Sch84], p. 47, Corollary 5.4, $\mathrm{Tr}_{L/K}(\langle 1 \rangle) = [L : K] \langle 1 \rangle$.

Now let L/K be an arbitrary finite separable field extension with $[\tilde{L} : K]$ odd. Then one has

$$\begin{aligned} [\tilde{L} : L] \langle 1 \rangle \cdot [L : K] \langle 1 \rangle &= [\tilde{L} : K] \langle 1 \rangle \\ &= \mathrm{Tr}_{\tilde{L}/K}(\langle 1 \rangle) \quad \text{as we have just shown} \\ &= \mathrm{Tr}_{L/K}(\mathrm{Tr}_{\tilde{L}/L}(\langle 1 \rangle)) \\ &= \mathrm{Tr}_{L/K}([\tilde{L} : L] \langle 1 \rangle) \quad \text{as we have just shown} \\ &= [\tilde{L} : L] \mathrm{Tr}_{L/K}(\langle 1 \rangle), \quad \text{since } \mathrm{Tr}_{L/K} \text{ is a homomorphism.} \end{aligned}$$

With [Sch84], p. 54, Corollary 6.5, the claim follows since $\dim([\tilde{L} : L] \langle 1 \rangle) = [\tilde{L} : L]$ is odd.

(ii): The first claim follows from [Sha79], p. 301, Theorem 5.5 and $\mathcal{N}(w(\langle 1 \rangle)) = \mathcal{N}(1) = 1$. If L/K is Galois of odd degree, the second claim follows from the first since the restriction is injective by [NSW00], p. 49, (1.5.7). In general, the second claim follows from (i) and $w([L : K] \langle 1 \rangle) = (1 + \{1\})^{[L:K]} = 1$. \square

Proposition 3.2. (i) *Let $L/M/K$ be finite separable field extensions such that the Galois hull \tilde{L}/M of L/M is of odd degree. Then one has $\mathrm{Tr}_{L/K}(\langle 1 \rangle) = [L : M] \mathrm{Tr}_{M/K}(\langle 1 \rangle)$.*

(ii) *Let $L/M/K$ be finite separable field extensions with $\mathrm{Tr}_{L/M}(\langle 1 \rangle) = \frac{[L:M]}{2} \langle 1, -1 \rangle$ totally hyperbolic. Then $\mathrm{Tr}_{L/K}(\langle 1 \rangle) = \frac{[L:K]}{2} \langle 1, -1 \rangle$ is totally hyperbolic.*

Proof. (i): One has

$$\begin{aligned} \mathrm{Tr}_{L/K}(\langle 1 \rangle) &= \mathrm{Tr}_{M/K}(\mathrm{Tr}_{L/M}(\langle 1 \rangle)) \\ &= \mathrm{Tr}_{M/K}([L : M] \langle 1 \rangle) \quad \text{by Proposition 3.1 (i)} \\ &= [L : M] \mathrm{Tr}_{M/K}(\langle 1 \rangle), \quad \text{since } \mathrm{Tr}_{M/K} \text{ is a homomorphism.} \end{aligned}$$

(ii) follows from the injectivity of the trace and [Lam05], p. 190, Corollary 1.4. \square

Proposition 3.3. *Let $L/K, L'/K$ be finite separable field extensions contained in a common field (so the compositum LL' is defined) and \tilde{L}/K the Galois hull of L/K . Assume $\tilde{L} \cap L' = K$.*

(i) *One has $\mathrm{Tr}_{LL'/L'}(\langle 1 \rangle_{LL'}) = r_{L'/K}^*(\mathrm{Tr}_{L/K}(\langle 1 \rangle_L))$.*

(ii) *One has $\mathrm{Tr}_{LL'/K}(\langle 1 \rangle_{LL'}) = \mathrm{Tr}_{L/K}(\langle 1 \rangle_L) \cdot \mathrm{Tr}_{L'/K}(\langle 1 \rangle_{L'})$.*

Proof. Because of $\tilde{L} \cap L' = K$, one has an isomorphism $LL' \cong L \otimes_K L'$ of L' -algebras, so the characteristic polynomial of $x^i \cdot -$ as a K -endomorphism of L equals the characteristic polynomial of $x^i \cdot -$ as an L' -endomorphism of LL' since the latter map is the base change by L' of the former map. In particular, $\text{Tr}_{L/K}(x^i x^j) = \text{Tr}_{LL'/L'}(x^i x^j)$, and one gets (i), since the x^i are a K -basis of L resp. a L' -basis of LL' for $0 \leq i < \deg(f)$.

Applying the projection formula [Sch84], p. 48, Theorem 5.6 to (i), one gets by the transitivity of the trace

$$\begin{aligned} \text{Tr}_{LL'/K}(\langle 1 \rangle_{LL'}) &= \text{Tr}_{L'/K}(\text{Tr}_{LL'/L'}(\langle 1 \rangle_{LL'})) \quad \text{because of the transitivity} \\ &= \text{Tr}_{L'/K}(r_{L'/K}^*(\text{Tr}_{L/K}(\langle 1 \rangle_L)) \cdot \langle 1 \rangle_{L'}) \quad \text{by (i)} \\ &= \text{Tr}_{L/K}(\langle 1 \rangle_L) \cdot \text{Tr}_{L'/K}(\langle 1 \rangle_{L'}) \quad \text{by the projection formula,} \end{aligned}$$

hence (ii). \square

Lemma 3.4. *Let (V, q) be a regular quadratic form over a field K , $W \subset V$ a totally isotropic K -subspace and $U \subset V$ a K -subspace with $W^\perp = W \oplus U$. Then there is a K -subspace W' of V with $\dim_K(W) = \dim_K(W')$, $W \oplus W'$ totally hyperbolic and $(W \oplus W') \perp U$.*

Proof. Let e_1, \dots, e_m be a K -basis of W . If $\dim_K(W) = 0$, one can take $W' = \{0\}$. So assume $m \geq 1$.

We inductively construct vectors e'_1, \dots, e'_m with $B_q(e_i, e'_j) = \delta_{ij}$ and $B_q(e'_i, U) = \{0\}$. Then $W' = \langle \{e'_1, \dots, e'_m\} \rangle$ works: One has $W \cap W' = \{0\}$ because since W is totally isotropic, one has $W \cap W' \subset W^\perp \cap W' = \{0\}$, as just shown. The e'_i are linearly independent since if $v = \sum \lambda_i e'_i = 0$, one has $0 = B_q(e_i, v) = \lambda_i$ for all i . Thus $W \oplus W'$ is totally hyperbolic by [Sch84], p. 12 f., Theorem 4.5 (with respect to the basis $e_1, \dots, e_m, e'_1, \dots, e'_m$ with $C = I_n$ because of $B_q(e_i, e'_j) = \delta_{ij}$ and the upper left square equal to 0 since W is totally isotropic; here, $W \oplus W'$ is regular by [Sch84], p. 7, Corollary 3.2 and [Lam05], p. 4, Proposition 1.2 since the matrix of B_q equals

$$\begin{pmatrix} 0 & I_n \\ I_n & * \end{pmatrix}$$

and is hence invertible). Further, one has $B_q(W \oplus W', U) \subset B_q(W, U) + B_q(W', U) = \{0\} + \{0\} = \{0\}$ since $U \subset W^\perp$ and $B_q(W', U)$ by construction of the e'_i . One has $(W \oplus W') \cap U = \{0\}$: For $v = \sum \lambda_i e_i + \sum \lambda'_i e'_i \in (W \oplus W') \cap U$, one has $0 = B_q(e_i, v) = \lambda'_i$ for all i because of $v \in U \subset W^\perp$ and since $B_q(e_i, e_j) = 0$ because of W totally isotropic and $B_q(e_i, e'_j) = \delta_{ij}$. Hence, $v = \sum \lambda_i e_i$. Thus, $0 = B_q(e'_i, v) = \lambda_i$ for all i because of $B_q(W', U) = \{0\}$ and $B_q(e'_i, e_j) = \delta_{ij}$, so $v = 0$. Summing up, one has $(W \oplus W') \perp U$.

Now for the construction of the e'_i .

Assume $U^\perp \subset \langle \{e_1\} \rangle^\perp$. Applying the inclusion-reversing $(-)^{\perp}$, it follows from [Sch84], p. 9, Lemma 3.11 and Corollary 3.12 that $\langle \{e_1\} \rangle \subset U$, a contradiction. Hence, there is $e'_1 \in U^\perp \setminus \langle \{e_1\} \rangle^\perp$. After division of e'_1 by $B_q(e_1, e'_1) \neq 0$ (since $e'_1 \notin \langle \{e_1\} \rangle^\perp$), one can assume $B_q(e_1, e'_1) = 1$. One has $B_q(e'_1, U) = \{0\}$ because of $e'_1 \in U^\perp$.

Assume that for $1 \leq i < i_0 \leq m$, the e'_i are already constructed. Let $U_{i_0} = \langle \{e_j : j < i_0\} \rangle \oplus U$. Assume $U_{i_0}^\perp \subset \langle \{e_{i_0}\} \rangle^\perp$. Applying the inclusion-reversing $(-)^{\perp}$, it follows from [Sch84], p. 9, Lemma 3.11 and Corollary 3.12 that $\langle \{e_{i_0}\} \rangle \subset U_{i_0}$, a contradiction. Hence, there is $e'_{i_0} \in U_{i_0}^\perp \setminus \langle \{e_{i_0}\} \rangle^\perp$. After division of e'_{i_0} by $B_q(e_{i_0}, e'_{i_0}) \neq 0$ (since $e'_{i_0} \notin \langle \{e_{i_0}\} \rangle^\perp$), one can assume $B_q(e_{i_0}, e'_{i_0}) = 1$. Further, one has $B_q(e_j, e'_{i_0}) = 0$ for $1 \leq j < i_0$ because of $e'_{i_0} \in U_{i_0}^\perp \subset \langle \{e_j : j < i_0\} \rangle^\perp$ and $B_q(e'_{i_0}, U) = \{0\}$ because of $e'_{i_0} \in U_{i_0}^\perp \subset U^\perp$. \square

Lemma 3.5. *Let $k \geq 1$ and K a field with $\text{char } K \nmid k$. Let*

$$\begin{aligned} f(X) &= \alpha \prod_{i=1}^m (X - \alpha_i) \in K[X], \\ g(X) &= \beta \prod_{i=1}^n (X - \beta_i) \in K[X] \end{aligned}$$

be two polynomials. Then

$$\text{res}(f(X^k), g(X^k)) = (-1)^{(k+1)mn} \text{res}(f(X), g(X))^k.$$

Proof. We calculate in the algebraic closure of K . Let ζ be a primitive k -th root of unity ($\text{char } K \nmid k$) and $\alpha_i^{\frac{1}{k}}, \beta_j^{\frac{1}{k}}$ fixed k -th roots of α_i, β_j . The factorisation of $f(X^k)$ in \bar{K} is

$$f(X^k) = \alpha \prod_{i=1}^m \prod_{a=1}^k (X - \zeta^a \alpha_i^{\frac{1}{k}})$$

and analogously for $g(X^k)$. Hence, by [Bos03], S. 177, Korollar 9

$$\begin{aligned} \text{res}(g(X^k), h(X^k)) &= \alpha^{kn} \beta^{km} \prod_{i=1}^m \prod_{j=1}^n \prod_{a,b=1}^k (\zeta^a \alpha_i^{\frac{1}{k}} - \zeta^b \beta_j^{\frac{1}{k}}) \\ &= \alpha^{kn} \beta^{km} \prod_{i=1}^m \prod_{j=1}^n \prod_{a,b=1}^k \zeta^a (\alpha_i^{\frac{1}{k}} - \zeta^{b-a} \beta_j^{\frac{1}{k}}) \\ &= \alpha^{kn} \beta^{km} \prod_{i=1}^m \prod_{j=1}^n \prod_{a=1}^k \zeta^a (\alpha_i - \beta_j) \\ &= (\alpha^n \beta^m)^k \prod_{i=1}^m \prod_{j=1}^n \zeta^{\frac{k(k+1)}{2}} (\alpha_i - \beta_j)^k \\ &= \zeta^{\frac{k(k+1)}{2} mn} \text{res}(g, h)^k, \end{aligned}$$

If k is odd, one has $\zeta^{\frac{k(k+1)}{2}} = (\zeta^k)^{\frac{k+1}{2}} = 1$, and if k is even, one has $\zeta^{\frac{k(k+1)}{2}} = (\zeta^{\frac{k}{2}})^{k+1} = (-1)^{k+1}$. Hence, the sign equals $(-1)^{(k+1)mn}$. \square

Lemma 3.6. *Let $k \geq 1$ and K be a field. Let $f(X) = X^n + \sum_{i=0}^{n-1} a_i X^i$ be a polynomial. Then the discriminant of $f(X^k)$ equals*

$$(-1)^{\frac{kn(kn-1)}{2}} (\text{res}(f, f'))^k k^{kn} a_0^{k-1}.$$

Proof. If $\text{char } K \mid k$, one has $(f(X^k))' = kX^{k-1}f'(X^k) = 0$ and hence its discriminant equals 0 by [Bos03], p. 173. So one can assume $\text{char } K \nmid k$.

By [Bos03], S. 177, Korollar 10, one has $\text{disc}(f(X^k)) = (-1)^{\frac{kn(kn-1)}{2}} \text{res}(f(X^k), (f(X^k))')$. We calculate in the algebraic closure of K . By [Bos03], S. 177, Korollar 9, one has

$$\begin{aligned} \text{res}(f(X^k), k) &= k^{kn}, \\ \text{res}(f(X^k), X^{k-1}) &= \prod_{f(\alpha)=0} \prod_{i=1}^k \left(\zeta^i \alpha^{\frac{1}{k}} \right)^{k-1} \\ &= \left(\prod_{f(\alpha)=0} \zeta^{\frac{k(k+1)}{2}} \alpha \right)^{k-1} \\ &= \left((-1)^{n(k+1)} \prod_{f(\alpha)=0} \alpha \right)^{k-1} \quad \text{because of } \zeta^{\frac{k(k+1)}{2}} = (-1)^{k+1} \\ &= (-1)^{n(k+1)(k-1)} ((-1)^n a_0)^{k-1} \\ &= (-1)^{n(k-1)} \cdot (-1)^{n(k-1)} \cdot a_0^{k-1} \quad \text{because of } k^2 - 1 \equiv k - 1 \pmod{2} \\ &= a_0^{k-1}, \end{aligned}$$

where ζ is a primitive k -th root of unity ($\text{char } K \nmid k$) and $\alpha^{\frac{1}{k}}$ is a fixed k -th root of α . The equality $\zeta^{\frac{k(k+1)}{2}} = (-1)^{k+1}$ is known from the proof of Lemma 3.5.

Hence,

$$\begin{aligned}
\operatorname{res}(f(X^k), (f(X^k))') &= \operatorname{res}(f(X^k), kX^{k-1}f'(X^k)) \\
&= \operatorname{res}(f(X^k), k) \operatorname{res}(f(X^k), X^{k-1}) \operatorname{res}(f(X^k), f'(X^k)) \\
&= k^{kn} a_0^{k-1} (-1)^{(k+1)n(n-1)} (\operatorname{res}(f, f'))^k \quad \text{by Lemma 3.5} \\
&= k^{kn} a_0^{k-1} (\operatorname{res}(f, f'))^k \quad \text{because of } n(n-1) \equiv 0 \pmod{2}
\end{aligned}$$

because of $\operatorname{res}(f, g_1 g_2) = \operatorname{res}(f, g_1) \operatorname{res}(f, g_2)$ (cf. [Bos03], S. 177, proof of Korollar 9).

Summing up, the claim follows. \square

Corollary 3.7. *Let $k \geq 1$ and K be a field. Let $n > 1$. Then the discriminant of $X^{kn} + a_k X^k + a_0 \in K[X]$ equals*

$$(-1)^{\frac{kn(kn-1)}{2}} [(1-n)^{n-1} a_k^n + n^n a_0^{n-1}]^k a_0^{k-1} k^{kn}.$$

Proof. For $k = 1$ see [Bos03], S. 181, Aufgabe 3 (ii). For the general case, set $f(X) = X^n + a_k X + a_0$ and apply Lemma 3.6. \square

Theorem 3.8. *Let $L = K(a)/K$ be a finite separable field extension of degree $n > 1$ with minimal polynomial $f(X) = X^n + \sum_{i=0}^{n-1} a_i X^i \in K[X]$ of a over K . Then the bilinear form associated to the trace form $\operatorname{Tr}_{L/K}(\langle 1 \rangle)$ with respect to the basis $1, a, a^2, \dots, a^{n-1}$ is given by the matrix $(a_{ij})_{i,j=0}^{n-1}$ with $a_{ij} = t_{i+j} = \operatorname{Tr}_{L/K}(a^{i+j})$, where the t_i are recursively determined by*

$$\begin{aligned}
t_0 &= n, \\
t_i &= - \left(i a_{n-i} + \sum_{j=1}^{i-1} a_{n-i+j} t_j \right), \quad i > 0.
\end{aligned}$$

Here $a_n = 1$ and $a_i = 0$ for $i < 0$. For $i > n$, the formula simplifies to

$$t_i = - \sum_{j=i-n}^{i-1} a_{n-i+j} t_j.$$

Proof. One has

$$\left(\sum_{i=0}^{n-1} x_i a^i \right)^2 = \sum_{i=0}^{n-1} x_i^2 a^{2i} + 2 \sum_{0 \leq i < j \leq n-1} x_i x_j a^{i+j}$$

for $x_i \in K$. Applying $\operatorname{Tr}_{L/K}$, one gets

$$\operatorname{Tr}_{L/K}(\langle 1 \rangle)(x_0, \dots, x_{n-1}) = \sum_{i=0}^{n-1} x_i^2 \operatorname{Tr}_{L/K}(a^{2i}) + 2 \sum_{0 \leq i < j \leq n-1} x_i x_j \operatorname{Tr}_{L/K}(a^{i+j}). \quad (3.1)$$

We show $\operatorname{Tr}_{L/K}(a^i) = t_i$.

Let A be the matrix of the K -endomorphism $L \rightarrow L, x \mapsto ax$ with respect to the basis $1, a, a^2, \dots, a^{n-1}$. Then $\operatorname{Tr}_{L/K}(a^i) = \operatorname{Tr}(A^i)$. Because of $f(A) = 0$, one has

$$\begin{aligned}
(X - A) \sum_{i=0}^{n-1} \left(\sum_{j=i}^{n-1} a_{j+1} A^{j-i} \right) X^i &= \sum_{i=1}^n \left(\sum_{j=i-1}^{n-1} a_{j+1} A^{j-i+1} \right) X^i - \sum_{i=0}^{n-1} \left(\sum_{j=i}^{n-1} a_{j+1} A^{j-i+1} \right) X^i \\
&= X^n - \sum_{j=0}^{n-1} a_{j+1} A^{j+1} + \sum_{i=1}^{n-1} a_i X^i \\
&= (f(X) - a_0) - (f(A) - a_0) \\
&= f(X).
\end{aligned}$$

Applying for $x \in \bar{K} \setminus \sigma(A)$ ($\sigma(A)$ the set of eigenvalues of A in \bar{K}) the homomorphism $M_n(K[X]) \rightarrow M_n(\bar{K})$, $X \mapsto xI_n = x$ and multiplying with $(x - A)^{-1}$ (x is no eigenvector of A) and taking the trace, yields

$$\sum_{i=0}^{n-1} \left(\sum_{j=i}^{n-1} a_{j+1} \operatorname{Tr}(A^{j-i}) \right) x^i = \operatorname{Tr}((x - A)^{-1} f(x)). \quad (3.2)$$

Let $T \in \operatorname{GL}_n(\bar{K})$ with TAT^{-1} an upper triangular matrix with diagonal entries x_i , where $f(x) = \prod_{i=1}^n (x - x_i)$. Then

$$\begin{aligned} \operatorname{Tr}((x - A)^{-1} f(x)) &= \operatorname{Tr}((x - A)^{-1}) f(x) \\ &= \operatorname{Tr}(T(x - A)^{-1} T^{-1}) f(x) \\ &= \operatorname{Tr}((x - TAT^{-1})^{-1}) f(x) \\ &= f(x) \sum_{i=1}^n (x - x_i)^{-1}. \end{aligned}$$

But

$$\begin{aligned} f(x) \sum_{i=1}^n (x - x_i)^{-1} &= \left(\prod_{j=1}^n (x - x_j) \right) \left(\sum_{i=1}^n (x - x_i)^{-1} \right) \\ &= \sum_{i=1}^n \prod_{j \neq i} (x - x_j) \\ &= f'(x). \end{aligned}$$

Plugging into (3.2), one gets

$$\sum_{i=0}^{n-1} \left(\sum_{j=i}^{n-1} a_{j+1} \operatorname{Tr}(A^{j-i}) \right) x^i = f'(x) = \sum_{i=0}^{n-1} (i+1) a_{i+1} x^i$$

for infinitely many $x \in \bar{K}$. One gets a polynomial with infinitely many zeroes, hence this equals 0, and comparing coefficients, one gets

$$\sum_{j=i}^{n-1} a_{j+1} \operatorname{Tr}(A^{j-i}) = (i+1) a_{i+1}$$

for all $0 \leq i \leq n-1$, which yields after an index shift

$$\sum_{j=1}^{n-i} a_{j+i} \operatorname{Tr}(A^{j-1}) = (i+1) a_{i+1} \quad (3.3)$$

for all $0 \leq i \leq n-1$.

For $i = n-1$, one gets from (3.3)

$$\operatorname{Tr}(A^0) = a_n \operatorname{Tr}(A^0) = n a_n = n = t_0.$$

Hence (3.3) simplifies to

$$n a_{i+1} + \sum_{j=2}^{n-i} a_{j+i} \operatorname{Tr}(A^{j-1}) = (i+1) a_{i+1},$$

so after subtraction of $n a_{i+1}$ and index shift

$$\operatorname{Tr}(A^{n-i}) + \sum_{j=1}^{n-i-1} a_{j+i} \operatorname{Tr}(A^j) = -(n-i) a_i \quad (3.4)$$

for all $1 \leq i \leq n$.

Now $\text{Tr}_{L/K}(a^i) = t_i$ for $i \leq n$ follows from induction: For $i = n - 2$, one gets from (3.4) $a_{n-1} \text{Tr}(A^0) + a_n \text{Tr}(A^1) = (n-1)a_{n-1}$, so

$$t_1 = -a_{n-1} = ((n-1) - n)a_{n-1} = (n-1)a_{n-1} - a_{n-1} \text{Tr}(A^0) = a_n \text{Tr}(A^1) = \text{Tr}(A^1).$$

Assume $\text{Tr}_{L/K}(a^i) = t_i$ is shown for $1 \leq i < i_0 \leq n$. Then

$$\begin{aligned} t_{i_0} &= - \left(i_0 a_{n-i_0} + \sum_{j=1}^{i_0-1} a_{n-i_0+j} t_j \right) \\ &\stackrel{\text{Vor.}}{=} - \left(i_0 a_{n-i_0} + \sum_{j=1}^{i_0-1} a_{n-i_0+j} \text{Tr}(A^j) \right) \\ &\stackrel{(3.4)}{=} - (i_0 a_{n-i_0} - \text{Tr}(A^{i_0}) - i_0 a_{n-i_0}) \\ &= \text{Tr}(A^{i_0}) \end{aligned}$$

using (3.4) with $0 \leq i = n - i_0 < n - 1$.

Assume $\text{Tr}_{L/K}(a^i) = t_i$ to be shown for all $i < i_0$ and $i_0 > n$. Then

$$\begin{aligned} 0 &= f(A)A^{n-i_0} \\ &= \sum_{j=0}^n a_j A^{n-i_0+j} \\ &= \sum_{j=i_0-n}^{i_0} a_{n-i_0+j} A^j \\ &= \sum_{j=i_0-n}^{i_0-1} a_{n-i_0+j} A^j + A^{i_0} \end{aligned}$$

and taking the trace and using the induction hypothesis yields $\text{Tr}_{L/K}(a^{i_0}) = t_{i_0}$. \square

Theorem 3.9. *Let $L = K(a)/K$ be a finite separable field extension of degree $kn > 1$ with minimal polynomial $f(X) = X^{kn} + a_k X^k + a_0 \in K[X]$ of a over K . Let*

$$c(f) = (1-n)^{n-1} a_k^n + n^n a_0^{n-1}.$$

Note that $\text{disc}(f) = (-1)^{\frac{kn(kn-1)}{2}} c(f)^k a_0^{k-1} k^{kn}$.

(1) Let $n = 2, k = 1$ and $\text{char } K \neq 2$. Then one has

$$\begin{aligned} \text{Tr}_{L/K}(\langle 1 \rangle) &= \langle 2, 2(a_1^2 - 4a_0) \rangle = \langle 2, 2 \text{disc}(f) \rangle \text{ and} \\ w(\text{Tr}_{L/K}(\langle 1 \rangle)) &= 1 + \{ \text{disc}(f) \} + \{ 2, -\text{disc}(f) \}. \end{aligned}$$

(2) Let $n > 2, k = 1$. Let $\text{char } K \nmid n$. If $a_1 = 0$ and $n > 2$, then

$$\text{Tr}_{L/K}(\langle 1 \rangle) = \begin{cases} \langle n + \frac{n-1}{2} \langle 1, -1 \rangle, & n \text{ odd} \\ \langle n, -na_0 \rangle + \frac{n-2}{2} \langle 1, -1 \rangle, & n \text{ even.} \end{cases}$$

(3) Let $k = 1, a_k \neq 0$ and $n > 2$. Let $\text{char } K \nmid n(n-1)$ and $a_1 \neq 0$ if n is odd. Then one has

$$\text{Tr}_{L/K}(\langle 1 \rangle) = \begin{cases} \left\langle n, (1-n)a_1, n(1-n)a_1(-1)^{\frac{n-3}{2}} \text{disc}(f) \right\rangle + \frac{n-3}{2} \langle 1, -1 \rangle, & n \text{ odd} \\ \left\langle n, (-1)^{\frac{n-2}{2}} n \text{disc}(f) \right\rangle + \frac{n-2}{2} \langle 1, -1 \rangle, & n \text{ even.} \end{cases}$$

(4) Let $n = 2, k > 1$ and $a_k \neq 0$. Let $\text{char } K \nmid 2k$. Then one has

$$\text{Tr}_{L/K}(\langle 1 \rangle) = \begin{cases} \langle 2k, -2kc(f), -ka_k, ka_k a_0 c(f) \rangle + (k-2) \langle 1, -1 \rangle, & k \text{ even} \\ \langle 2k, -2kc(f) \rangle + (k-1) \langle 1, -1 \rangle, & k \text{ odd.} \end{cases}$$

(5) Let $n > 2, k > 1$ and $a_k \neq 0$. Let $\text{char } K \nmid kn(n-1)$. Then one has $\text{Tr}_{L/K}(\langle 1 \rangle) =$

$$\begin{cases} \langle kn, k(1-n)a_k, -a_0 kc(f), -n(1-n)a_k kc(f) \rangle + (\frac{k}{2}n-2) \langle 1, -1 \rangle, & k \text{ even, } n \text{ odd} \\ \langle kn, k(1-n)a_k, -ka_0 a_k, -kn(1-n) \rangle + (\frac{k}{2}n-2) \langle 1, -1 \rangle, & k \text{ even, } n \text{ even} \\ \langle kn, -c(f)kn \rangle + \frac{kn-2}{2} \langle 1, -1 \rangle, & k \text{ odd, } n \text{ even} \\ \langle kn, k(1-n)a_k, -c(f)kn(1-n)a_k \rangle + (\frac{kn-3}{2}) \langle 1, -1 \rangle, & k \text{ odd, } n \text{ odd.} \end{cases}$$

Proof. Denote the associated bilinear form to the trace form $\text{Tr}_{L/K}(\langle 1 \rangle)$ with respect to the basis $1, a, a^2, \dots, a^{n-1}$ by the matrix $(a_{ij})_{i,j=0}^{n-1}$ with $a_{ij} = t_{i+j} = \text{Tr}_{L/K}(a^{i+j})$. One first calculates from Theorem 3.8:

(i) Let $n = 2, k = 1$. Then one has

$$\begin{aligned} t_0 &= kn = 2, \\ t_1 &= -k(n-1)a_k = -a_1, \\ t_2 &= k(-2a_0 + a_k^2) = a_1^2 - 2a_0. \end{aligned}$$

(ii) Let $n = 2, k > 1$. Then one has

$$\begin{aligned} t_0 &= kn = 2k, \\ t_k &= t_{k(n-1)} = -k(n-1)a_k = -ka_k, \\ t_{2k} &= t_{kn} = k(a_k^2 - 2a_0), \\ t_{3k} &= t_{k(2n-1)} = ka_k((n+1)a_0 - a_k^2) = ka_k(3a_0 - a_k^2). \end{aligned}$$

(iii) Let $n > 2, k = 1$. Then one has

$$\begin{aligned} t_0 &= kn = n, \\ t_{n-1} &= t_{k(n-1)} = k(1-n)a_k = (1-n)a_1, \\ t_n &= t_{kn} = -kna_0 = -na_0, \\ t_{2(n-1)} &= t_{2(kn-1)} = k(n-1)a_k^2 = (n-1)a_1^2. \end{aligned}$$

(iv) Let $n > 2$ and $k \geq 2$. Then one has

$$\begin{aligned} t_0 &= kn, \\ t_{k(n-1)} &= k(1-n)a_k, \\ t_{kn} &= -kna_0, \\ t_{k(2n-2)} &= k(n-1)a_k^2, \\ t_{k(2n-1)} &= k(2n-1)a_0 a_k. \end{aligned}$$

Then one diagonalises the symmetric matrix $(a_{ij})_{i,j=0}^{n-1}$:

By example, we show (5) since this is the most complicated case. We distinguish the cases k even and k odd. By example, we show the case k even. We distinguish the cases n odd resp. even. We show by example the case n odd.

Let $W \subset L$ the K -subspace spanned by $a^1, \dots, a^{\frac{k}{2}(n-1)-1}$. Because of $0 < 2 \cdot 1 \leq 2(\frac{k}{2}(n-1)-1) < k(n-1)$ and $t_i = 0$ for $0 < i < k(n-1)$ by the calculation from (iv), one has $B_q(a^i, a^j) = \text{Tr}(a^{i+j}) = t_{i+j} = 0$ for $1 \leq i, j \leq \frac{k}{2}(n-1)-1$, so $B_q(W, W) = \{0\}$. Let $U = \langle \{1, a^{\frac{k}{2}(n-1)}, v_1, \dots, v_k\} \rangle$ with

$$\begin{aligned} \lambda &= -\frac{\text{Tr}(a^{kn})}{\text{Tr}(a^{k(n-1)})} = \frac{na_0}{(1-n)a_k}, \\ v_j &= \sum_{i=0}^{i_{\max}(j,k,n)} \lambda^i a^{k(n-i)-j}, j = 1, \dots, k. \end{aligned}$$

Here, let $i_{\max}(j, k, n)$ be the maximal $i \geq 0$ with $k(n - i) - j \geq \frac{k}{2}(n - 1) + 1$. One has

$$i_{\max}(j, k, n) = \left\lfloor \frac{n+1}{2} - \frac{j+1}{k} \right\rfloor.$$

Now assume n odd. Then, $\frac{n+1}{2} \in \mathbf{N}$ and hence

$$i_{\max}(j, k, n) = \frac{n+1}{2} - \left\lfloor \frac{j+1}{k} \right\rfloor.$$

Thus, $i_{\max}(j, k, n) = \frac{n-1}{2}$ for $1 \leq j \leq k-1$ and $i_{\max}(j, k, n) = \frac{n-3}{2}$ for $j = k$, and hence $k(n - i_{\max}(j, k, n)) - j = k \cdot \frac{n+1}{2} - j$ for $1 \leq j \leq k-1$ and $k(n - i_{\max}(j, k, n)) - j = k \cdot \frac{n+1}{2}$ for $j = k$.

Because of $0 < 1 \leq \frac{k}{2}(n-1) - 1 < k(n-1)$ and $0 < 1 + \frac{k}{2}(n-1) \leq \frac{k}{2}(n-1) - 1 + \frac{k}{2}(n-1) = k(n-1) - 1 < k(n-1)$, one has with (iv) $\langle \{1, a^{\frac{k}{2}(n-1)}\} \rangle \subset W^\perp$. Let $(j = 1, \dots, k, m = 1, \dots, \frac{k}{2}(n-1) - 1)$ or $(m = 0, \frac{k}{2}(n-1)$ and $1 \leq j \leq \frac{k}{2})$. One has

$$\mathrm{Tr}(a^m v_j) = \sum_{i=0}^{i_{\max}(j, k, n)} \lambda^i \mathrm{Tr}(a^{k(n-i)-j+m}). \quad (3.5)$$

We want to show that this always equals 0. If $-j + m = kr$ (note that $r \geq 0$ because of $-j + m \geq -k + 1 > -k$ bzw. $-j + m \geq -\frac{k}{2} + 0 > -k$) the sum from (3.5) equals

$$\begin{aligned} & \lambda^r (\mathrm{Tr}(a^{kn}) + \lambda \mathrm{Tr}(a^{k(n-1)})) = \\ & \lambda^r \left(\mathrm{Tr}(a^{kn}) - \frac{\mathrm{Tr}(a^{kn})}{\mathrm{Tr}(a^{k(n-1)})} \cdot \mathrm{Tr}(a^{k(n-1)}) \right) = \\ & \lambda^r \cdot 0 = 0, \end{aligned}$$

since in the sum (3.5), the exponent of a is at least equal to $k(n - \frac{n-1}{2}) - j + m = \frac{k}{2}(n+1) - j + m > \frac{k}{2}(n+1) - k = \frac{k}{2}(n-1) > 0$ and at most equal to $kn - 1 + \frac{k}{2}(n-1) = \frac{3}{2}kn - \frac{k}{2} - 1 = \frac{k}{2}(3n-1) - 1 < k(2n-2)$ as the latter inequality is equivalent to $-1 < \frac{k}{2}(n-3)$, which is satisfied for $n \geq 3$. So by (iv), only terms with $\mathrm{Tr}(a^{k(n-1)})$ and $\mathrm{Tr}(a^{kn})$ can be $\neq 0$. Both indeed occur since $i_{\max}(j, k, n) = \frac{n-1}{2} > 0$ for $1 \leq j \leq k-1$, and for $j = k$ one has $i_{\max}(j, k, n) = \frac{n-3}{2} > 0$ for $n > 3$. In the case $n = 3, j = k$ one has $1 \leq m \leq k-1$ and hence $k+1 \leq -j + m \leq k + (k-1) = 2k-1$, so $-j + m = kr$ cannot occur.

In the case $-j + m \not\equiv 0 \pmod{k}$, in the sum (3.5) only terms $\mathrm{Tr}(a^s)$ with $k \nmid s$ occur, so by (iv) all terms equal 0.

Summing up, one gets $U \subset W^\perp$.

The generators of U from above are linearly independent since the exponents of a in the sums for the v_j lie by definition of $i_{\max}(j, k, n)$ between $kn - 1$ and $\frac{k}{2}(n-1) + 1$ and for different $1 \leq j \leq k$ the exponents of a in the sum are all distinct since they are $\equiv -j \pmod{k}$ and the a^i for $0 \leq i \leq kn - 1$ are a K -basis of L .

The sum $W + U$ is direct, and because of $\frac{k}{2}(n-1) - 1 + \dim_K(W^\perp) = \dim_K(W) + \dim_K(W^\perp) = kn$ ([Sch84], p. 9, Lemma 3.11 and Corollary 3.12) one has $\dim_K(W^\perp) = kn - (\frac{k}{2}(n-1) - 1) = \frac{k}{2}(n+1) + 1$. But since W is totally isotropic, one has $W \subset W^\perp$ und damit $W \oplus U \subset W^\perp$ with $\dim_K(W \oplus U) = (\frac{k}{2}(n-1) - 1) + (k+2) = \frac{k}{2}(n+1) + 1 = \dim_K(W^\perp)$, hence $W \oplus U = W^\perp$.

Let $W' \subset L$ as in Lemma 3.4. ($\mathrm{Tr}(\langle 1 \rangle)$ is regular.) Then one has $\dim_K((W \oplus W') \perp U) = 2 \cdot (\frac{k}{2}(n-1) - 1) + (k+2) = kn$, so $L = (W \oplus W') \perp U$. Hence $\mathrm{Tr}(\langle 1 \rangle)$ equals $(\frac{k}{2}(n-1) - 1) \cdot \langle 1, -1 \rangle$ plus the trace form restricted to U . Now we diagonalise the latter.

Let $W_1 \subset U$ be the K -subspace spanned by $v_1, \dots, v_{k/2-1}$. One has $B_q(W_1, W_1) = \{0\}$ nach den Berechnungen von (iv): Writing $v_i v_j$ in the basis a^0, \dots, a^{kn-1} , only powers of a with exponents congruent to $-i - j \pmod{k}$ occur. Because of $1 \leq i, j \leq \frac{k}{2} - 1$ all exponents are $\not\equiv 0 \pmod{k}$. Hence, by (iv) the trace of $v_i v_j$ equals 0.

Because of $\frac{k}{2} - 1 + \dim_K(W_1^\perp) = \dim_K(W_1) + \dim_K(W_1^\perp) = \dim_K(U) = k + 2$ ([Sch84], p. 9, Lemma 3.11 and Corollary 3.12) one has $\dim_K(W_1^\perp) = k + 2 - (\frac{k}{2} - 1) = \frac{k}{2} + 3$. But since W_1 is totally isotropic, one has $W_1 \subset W_1^\perp$ and hence $\dim_K(W_1^\perp/W_1) = (\frac{k}{2} + 3) - (\frac{k}{2} - 1) = 4$.

By (iv), one has (since all exponents of a are between $2(kn - \frac{k}{2}) = k(2n - 1)$ and $2(k \cdot \frac{n+1}{2} - \frac{k}{2}) = kn$)

$$\begin{aligned}
q(v_{k/2}) &= \text{Tr} \left(\left[\sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \lambda^i a^{k(n-i) - \frac{k}{2}} \right]^2 \right) \\
&= \text{Tr}(a^{k(2n-1)}) + 2\lambda \text{Tr}(a^{k(2n-2)}) + \lambda^{n-1} \text{Tr}(a^{kn}) \\
&= k(2n-1)a_0 a_k + 2\lambda \cdot k(n-1)a_k^2 - \lambda^{n-1} k n a_0 \\
&= k(2n-1)a_0 a_k + 2 \frac{na_0}{(1-n)a_k} k(n-1)a_k^2 - \left(\frac{na_0}{(1-n)a_k} \right)^{n-1} k n a_0 \\
&= -a_0 k \left[a_k + \left(\frac{na_0}{(1-n)a_k} \right)^{n-1} n \right] \\
&\equiv -a_0 k [(1-n)^{n-1} a_k^n + n^n a_0^{n-1}] \pmod{(K^\times)^2} \\
&= -a_0 k c(f).
\end{aligned}$$

In the last step we used that since n is odd, $[(1-n)a_k]^{n-1}$ is a square. The latter expression is $\neq 0$ because of $\text{char } K \nmid k$ and Corollary 3.7.

Let $\tilde{U}_1 = \langle \{1, a^{\frac{k}{2}(n-1)}, v_{k/2}\} \rangle$. One has $\tilde{U}_1 \subset W_1^\perp$ because of $1, a^{\frac{k}{2}(n-1)} \in W_1^\perp$ and $v_{k/2} \in W_1^\perp$ since for $1 \leq i \leq \frac{k}{2} - 1$ one has $-\frac{k}{2} - i \not\equiv 0 \pmod{k}$, and hence $B_q(v_{k/2}, W_1) = \{0\}$.

The vectors form an orthogonal basis of \tilde{U}_1 since $\text{Tr}(1 \cdot a^{\frac{k}{2}(n-1)}) = 0 = \text{Tr}(1 \cdot v_{k/2})$ as all exponents of a are $\not\equiv 0 \pmod{k}$, and $\text{Tr}(a^{\frac{k}{2}(n-1)} \cdot v_{k/2}) = 0$.

Let $0 \neq \tilde{v} \in W_1^\perp$ such that $(\tilde{U}_1 \oplus \langle \{\tilde{v}\} \rangle) \oplus W_1 = W_1^\perp$. (it exists since $\dim_K(W_1^\perp / (\tilde{U}_1 \oplus W_1)) = (\frac{k}{2} + 3) - (3 + \frac{k}{2} - 1) = 1$). Let $U_1 = \tilde{U}_1 \oplus \langle \{\tilde{v}\} \rangle$. By construction, one has $U_1 \oplus W_1 = W_1^\perp$. Let $v \in U_1$ be the vector obtained by applying the Gram-Schmidt orthogonalisation to \tilde{v} with respect to $1, a^{\frac{k}{2}(n-1)}, v_{k/2}$. This is possible since $q(1), q(a^{\frac{k}{2}(n-1)}), q(v_{k/2})$ are all $\neq 0$.

Let $W'_1 \subset U$ as in Lemma 3.4 (applied to $(q|_U, U)$ with $W_1^\perp = W_1 \oplus U_1$. $(U, q|_U)$ is regular: (L, q) is regular and $L = U \perp (W \oplus W')$. Now apply [Sch84], p. 10, Lemma 3.14 (iii).) Hence $q|_U$ equals $(\frac{k}{2} - 1) \cdot \langle 1, -1 \rangle$ plus the trace form restricted to U_1 .

Summing up, one has $q = (W \oplus W') \perp (W_1 \oplus W'_1) \perp U_1 = (\frac{k}{2}(n-1) - 1) \langle 1, -1 \rangle + (\frac{k}{2} - 1) \langle 1, -1 \rangle + q|_{U_1} = (\frac{k}{2}n - 2) \langle 1, -1 \rangle + q|_{U_1}$. It remains to calculate $q|_{U_1}$.

Since the basis vectors from above together with v form an orthogonal basis, one obtains

$$\begin{aligned}
q|_{U_1} &= \left\langle q(1), q(a^{\frac{k}{2}(n-1)}), q(v_{k/2}), q(v) \right\rangle \\
&= \left\langle kn, k(1-n)a_k, q(v_{k/2}), (-1)^{\frac{k}{2}n-2} \frac{\text{disc}(f)}{n(1-n)a_k \cdot q(v_{k/2})} \right\rangle \\
&= \left\langle kn, k(1-n)a_k, q(v_{k/2}), (-1)^{\frac{k}{2}n-2} \frac{(-1)^{\frac{k}{2}} a_0}{n(1-n)a_k \cdot q(v_{k/2})} \right\rangle \\
&= \left\langle kn, k(1-n)a_k, q(v_{k/2}), (-1)^{\frac{k}{2}(n+1)-1} \frac{a_0}{n(1-n)a_k a_0 k c(f)} \right\rangle \\
&= \langle kn, k(1-n)a_k, -a_0 k c(f), -n(1-n)a_k k c(f) \rangle.
\end{aligned}$$

Here, we used that if $\text{Tr}_{L/K}(\langle 1 \rangle) = \langle a_1, \dots, a_{n-1}, a_n \rangle$, $a_n \equiv \text{disc}(f) / \prod_{i=1}^{n-1} a_i \pmod{(K^\times)^2}$ ([Bos03], S. 179 f., Satz 11 and [Sch84], p. 37, Remark 2.2), $\frac{k}{2}(n+1) \equiv 0 \pmod{2}$ und $\text{disc}(f) \equiv (-1)^{\frac{k}{2}} a_0 \pmod{(K^\times)^2}$ by Corollary 3.7 because of k even, n odd. \square

Margit Angermeier [Ang13] and Ramona Schiller [Sch13] extend in their master theses the above work to minimal polynomials of the form

$$X^{kn} + a_{2k} X^{2k} + a_k X^k + a_0$$

for $k \geq 1$ and $n > 2$.

From these calculations, one easily derives their Stiefel-Whitney invariants. In the case (2) of Theorem 3.9, if $\mu_n \subset K$, then L/K is normal, and for n odd together with the fact that the trace form of a Galois extension of odd degree $[L : K]$ is equal to $[L : K] \langle 1 \rangle$ by Proposition 3.1, one gets relations in $K_*^M(K)/2$:

$$\begin{aligned} 1 &= w(\mathrm{Tr}_{L/K}(\langle 1 \rangle)) \\ &= w(\{n\} + \frac{n-1}{2} \langle 1, -1 \rangle) \\ &= 1 + \{(-1)^{\frac{n-1}{2}} n\} + \sum_{i=2}^{(n+1)/2} \{(-1)^{\binom{(n-1)/2}{i}} n^{\binom{(n-1)/2}{i-1}}\} \cdot \{-1\}^{i-1}. \end{aligned}$$

For example, it follows that $\sqrt{(-1)^{(n-1)/2} n} \in K$ (this result is usually proven using Gauss sums).

In the top degree $\frac{n+1}{2}$ it follows that $\{n\} \cdot \{-1\}^{(n-1)/2} = 0$. The relations have the following well-known interpretation for quadratic forms, which we recall for convenience of the reader.

Proposition 3.10. *Let $n > 1$, $a_1, \dots, a_n \in K^\times$. The following are equivalent:*

- (i) $\{a_1, \dots, a_n\} = 0 \in K_n^M(K)/2$
- (ii) $\langle\langle a_1, \dots, a_n \rangle\rangle$ is totally hyperbolic.
- (iii) $\langle\langle a_1, \dots, a_n \rangle\rangle$ is isotropic.
- (iv) a_n is represented by $\langle\langle a_1, \dots, a_{n-1} \rangle\rangle$.

Proof. (i) \iff (ii): By the Milnor conjecture on quadratic forms [OVV00] (note that this is easier for $n = 2$ and already proved in [Mil70]; furthermore, we only need injectivity on *pure* symbols), one has $\{a_1, \dots, a_n\} = 0 = \{1, \dots, 1\}$ if and only if $\langle\langle a_1, \dots, a_n \rangle\rangle \equiv \langle\langle 1, \dots, 1 \rangle\rangle = 2^{n-1} \langle 1, -1 \rangle \pmod{I^{n+1}}$. This is by [Lam05], p. 353, Corollary 5.4 equivalent to $\langle\langle a_1, \dots, a_n \rangle\rangle \equiv \langle\langle 1, \dots, 1 \rangle\rangle = 0 \in W(K)$.

(ii) \iff (iii): a totally hyperbolic form is isotropic, and for the converse see [Lam05], p. 319, Theorem 1.7.

(iii) \implies (iv): By assumption, there is $0 \neq (v_1, v_2) \in K^{2^{n-1}} \times K^{2^{n-1}}$ with

$$\langle\langle a_1, \dots, a_{n-1} \rangle\rangle(v_1) - a_n \langle\langle a_1, \dots, a_{n-1} \rangle\rangle(v_2) = 0.$$

If $\langle\langle a_1, \dots, a_{n-1} \rangle\rangle(v_1)$ or $\langle\langle a_1, \dots, a_{n-1} \rangle\rangle(v_2)$ equals 0, $\langle\langle a_1, \dots, a_{n-1} \rangle\rangle$ is isotropic (since v_1, v_2 are not both 0), hence totally hyperbolic by [Lam05], p. 319, Theorem 1.7, and thus represents a_n . Otherwise, there is $v_3 \in K^{2^{n-1}}$ with

$$\langle\langle a_1, \dots, a_{n-1} \rangle\rangle(v_3) = \frac{\langle\langle a_1, \dots, a_{n-1} \rangle\rangle(v_1)}{\langle\langle a_1, \dots, a_{n-1} \rangle\rangle(v_2)} = a_n,$$

since Pfister forms are multiplicative by [Sch84], p. 144, Corollary 1.5 (if a form q is anisotropic, one has $D(q) = G(q)$ with the notation from p. 142, and $G(q)$ is obviously a group; if q is hyperbolic, $D(q) = K^\times$ by [Sch84], p. 12 f., Theorem 4.5). Hence, in all cases $\langle\langle a_1, \dots, a_{n-1} \rangle\rangle$ represents a_n .

(iii) \iff (iv): By assumption, there is $v \in K^{2^{n-1}}$ with $a_n = \langle\langle a_1, \dots, a_{n-1} \rangle\rangle(v)$ (denote $\langle\langle a_1, \dots, a_{n-1} \rangle\rangle(v) = q(v)$ by $q = \langle\langle a_1, \dots, a_{n-1} \rangle\rangle$), so

$$\begin{aligned} \langle\langle a_1, \dots, a_n \rangle\rangle(v, (1, 0, \dots)) &= \langle\langle a_1, \dots, a_{n-1} \rangle\rangle(v) - a_n \langle\langle a_1, \dots, a_{n-1} \rangle\rangle(1, 0, \dots) \\ &= a_n - a_n \\ &= 0. \end{aligned} \quad \square$$

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TIMO KELLER
 MATHEMATIK II
 COMPUTERALGEBRA
 UNIVERSITÄT BAYREUTH
 UNIVERSITÄTSSTRASSE 30
 95440 BAYREUTH
 DEUTSCHLAND
 e-mail address: Timo.Keller@uni-bayreuth.de