On an analogue of the conjecture of Birch and Swinnerton-Dyer for Abelian schemes over higher dimensional bases over finite fields – complements

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Abstract

We prove a finiteness theorem for the first flat cohomology group of finite flat group schemes over integral normal proper varieties over finite fields. As a consequence, we can prove the invariance of the finiteness of the Tate-Shafarevich group of Abelian schemes over higher dimensional bases under isogenies and alterations over/of such bases for the p-part. Along the way, we generalize previous results on the Tate-Shafarevich and the Brauer group in this situation.

Keywords: Abelian varieties of dimension > 1; Birch-Swinnerton-Dyer conjecture; Étale and other Grothendieck topologies and cohomologies; Brauer groups of schemes; Arithmetic ground fields

MSC 2010: 11G10, 14F20, 14F22, 14K15

1 Introduction

The Tate-Shafarevich group $Ш(\mathscr{A}/X)$ of an Abelian scheme $\mathscr{A}$ over a base scheme X is of great importance for the arithmetic of $\mathscr{A}$. It classifies everywhere locally trivial $\mathscr{A}$-torsors. Its finiteness is sufficient to establish our analogue of the conjecture of Birch and Swinnerton-Dyer [Kel19] over higher dimensional bases over finite fields.

In [Kel16, section 4.3], we showed that finiteness of an ℓ-primary component of the Tate-Shafarevich group descends under generically étale ℓ'-alterations. This is used in [Kel19, Corollary 5.11] to prove the finiteness of the Tate-Shafarevich group and an analogue of the Birch-Swinnerton-Dyer conjecture for certain Abelian schemes over higher dimensional bases over finite fields under mild conditions. In [Kel16, section 4.4], we showed that finiteness of an ℓ-primary component of the Tate-Shafarevich group is invariant under étale isogenies.

In this article, we generalize these results for ℓ^∞-torsion to p^∞-torsion and complete our reduction from a basis of dimension 2 to a basis of dimension 1 assuming the ground field is an infinite finitely generated field of positive characteristic.

Recall that we defined the Tate-Shafarevich group of $\mathscr{A}/X$ for $\dim X > 1$ and $\mathscr{A}$ of good reduction as $\mathrm{H}^1_{\text{ét}}(X, \mathscr{A})$. We first weaken the hypothesis that $\mathscr{A}$ is a Jacobian in [Kel16, Theorem 4.5] to arbitrary Abelian schemes:

Theorem (Corollary 2.4). *Let X be a regular integral Noetherian separated scheme and $\mathscr{A}/X$ be an Abelian scheme. Let $Z \hookrightarrow X$ be a closed subscheme of codimension ≥ 2. Then $\mathrm{H}^i_Z(X, \mathscr{A})$ is torsion for all i and has trivial p-torsion for $i = 0, 1, 2$ and if p is invertible on X.*

Our main results are now as follows:

Theorem (Theorem 3.14). *Let X be a proper integral normal variety over a finite field and G/X be a finite flat commutative group scheme. Then $\mathrm{H}^1_{\text{fppf}}(X, G)$ is finite.*

This theorem is proven by reduction to the finite flat simple group schemes $\mathbf{Z}/p$,μ_p and α_p over an algebraically closed field using by de Jong's alteration theorem, Raynaud-Gruson and a dévissage argument.

Using this technical result and refining our methods from [Kel16], we obtain:

Theorem (Lemma 5.1)**.** *Let $\mathscr{A}/X$ be an Abelian scheme over a proper variety X over a finite field of characteristic p. Then $Ш(\mathscr{A}/X)[p^\infty]$ is cofinitely generated.*

Theorem (invariance of finiteness of Ш under isogenies)**.** *Let X/k be a proper variety over a finite field k and $f : \mathscr{A} \to \mathscr{A}'$ be an isogeny of Abelian schemes over X. Let p be an arbitrary prime. Assume f étale if $p \neq \operatorname{char} k$. Then $Ш(\mathscr{A}/X)[p^\infty]$ is finite if and only if $Ш(\mathscr{A}'/X)[p^\infty]$ is finite.*

Theorem (invariance of finiteness of Ш under alterations)**.** *Let $f : X' \to X$ be a proper, surjective, generically finite morphism of integral, normal varieties over a finite field. Let $\mathscr{A}$ be an abelian scheme on X and $\mathscr{A}' := f^*\mathscr{A} = \mathscr{A} \times_X X'$. Let ℓ be invertible on X. If $Ш(\mathscr{A}'/X')[\ell^\infty]$ is finite, so is $Ш(\mathscr{A}/X)[\ell^\infty]$.*

Notation. Canonical isomorphisms are often denoted by "=". The ℓ-adic valuation $|\cdot|_\ell$ is taken to be normalized by $|\ell|_\ell = \ell^{-1}$. Denote the absolute Galois group of a field k by G_k.

We denote Pontrjagin duality by $(-)^D$ (see [NSW00, § 1]), duals of R-modules or ℓ-adic sheaves by $(-)^\vee$, and duals of Abelian schemes and Cartier duals by $(-)^t$.

For an abelian group A, let A_{tors} be the torsion subgroup of A, and $A_{\text{n-tors}} = A/A_{\text{tors}}$. Let A_{div} be the maximal divisible subgroup of A (in general strictly contained in the subgroup of divisible elements of A, but agree for A a cofinitely generated ℓ-primary group [Kel19, Lemma 2.1.1 (iii)]) and $A_{\text{n-div}} = A/A_{\text{div}}$. For an integer n and an object A of an abelian category, denote the cokernel of $A \xrightarrow{n} A$ by A/n and its kernel by $A[n]$, and for a prime p the p-primary subgroup $\varinjlim_n A[p^n]$ by $A[p^\infty]$. Write $A[\text{non-}p]$ or $A[p']$ for $\varinjlim_{p\nmid n} A[n]$. For a prime ℓ, let the ℓ-adic Tate module $T_\ell A$ be $\varprojlim_n A[\ell^n]$ and the rationalized ℓ-adic Tate module $V_\ell A = T_\ell A \otimes_{\mathbf{Z}_\ell} \mathbf{Q}_\ell$. The corank of $A[p^\infty]$ is the $\mathbf{Z}_p$-rank of $A[p^\infty]^D = T_p A$.

2 Vanishing of étale cohomology with supports of Abelian schemes

This is a complement to the "vanishing condition" $\mathrm{H}^i_Z(X, G) = 0$ [Kel16, (4.4)], which is proven there only for Jacobians of curves, see [Kel16, Lemma 4.10].

Theorem 2.1. *Let X be a regular integral Noetherian separated scheme and G/X be a finite étale commutative group scheme of order invertible on X. Let $Z \hookrightarrow X$ be a closed subscheme of codimension ≥ 2. Then $\mathrm{H}^i_Z(X, G) = 0$ for $i \leq 2$ (étale cohomology with supports in Z).*

Proof. Let $U = X \setminus Z$. One has a long exact cohomology sequence

$$\ldots \to \mathrm{H}^{i-1}(X, G) \to \mathrm{H}^{i-1}(U, G) \to \mathrm{H}^i_Z(X, G) \to \mathrm{H}^i(X, G) \to \mathrm{H}^i(U, G) \to \ldots,$$

so one has to prove that $\mathrm{H}^i(X, G) \to \mathrm{H}^i(U, G)$ is an isomorphism for $i = 0, 1$ and injective for $i = 2$.

For $i = 0$, the claim $\mathrm{H}^i_Z(X, G) = 0$ is equivalent to the injectivity of

$$\mathrm{H}^0(X, G) \to \mathrm{H}^0(X \setminus Z, G),$$

which is clear from [Har83], p. 105, Exercise II.4.2 since G/X is separated, X is reduced and $X \setminus Z \hookrightarrow X$ is dense.

For $i = 1$ the claim $\mathrm{H}^i_Z(X, G) = 0$ is equivalent to

$$\mathrm{H}^0(X, G) \to \mathrm{H}^0(X \setminus Z, G)$$

being surjective and

$$\mathrm{H}^1(X, G) \to \mathrm{H}^1(X \setminus Z, G)$$

being injective. The surjectivity of $\mathrm{H}^0(X, G) \to \mathrm{H}^0(X \setminus Z, G)$ follows e. g. from

Theorem 2.2. *Let S be a normal Noetherian base scheme, and let $u : Z \dashrightarrow G$ be an S-rational map from a smooth S-scheme Z to a smooth and separated S-group scheme G. Then, if u is defined in codimension ≤ 1, it is defined everywhere.*

Proof. See [BLR90], p. 109, Theorem 1. □

For the injectivity of $\mathrm{H}^1(X,G) \to \mathrm{H}^1(X \setminus Z, G)$: If a principal homogeneous space P/X for G/X is trivial over $X \setminus Z$, then it is trivial over X: The trivialization over $X \setminus Z$ gives a rational map from X to the principal homogeneous space and any such map (with X a regular scheme) extends to a morphism by Theorem 2.2.

For the surjectivity of $\mathrm{H}^1(X,G) \to \mathrm{H}^1(X \setminus Z, G)$: This means that any principal homogeneous space $P/(X \setminus Z)$ extends to a principal homogeneous space $\bar{P}/X$. By [Mil80], p. 123, Corollary III.4.7, we have $\mathrm{PHS}(G/X) \xrightarrow{\sim} \mathrm{H}^1(X_{\mathrm{fl}}, G)$ (Čech cohomology) since G/X is affine. Since G/X is smooth, [Mil80], p. 123, Remark III.4.8 (a) shows that we can take étale cohomology as well, and by [Mil80], p. 101, Corollary III.2.10, one can take derived functor cohomology instead of Čech cohomology. By Zariski-Nagata purity [SGA1], Exp. X, Corollaire 3.3, one can extend this to a $\bar{P}/X$, for which we have to show that it represents an element of $\mathrm{H}^1(X,G)$, i. e. that it is a G-torsor.

Now we need to show that if $P/(X \setminus Z)$ is an $G|_{X\setminus Z}$-torsor and $\bar{P}$ an extension of P to a finite étale covering of X, then $\bar{P}/X$ is also an G-torsor. For this, we use the following

Theorem 2.3. *Let S be a connected scheme, $G \to S$ a finite flat group scheme, and $X \to S$ a scheme over S equipped with a left action $\rho : G \times_S X \to X$. These data define a G-torsor over S if and only if there exists a finite locally free surjective morphism $Y \to S$ such that $X \times_S Y \to Y$ is isomorphic, as a Y-scheme with $G \times_S Y$-action, to $G \times_S Y$ acting on itself by left translations.*

Proof. See [Sza09], p. 171, Lemma 5.3.13. □

That $P/(X \setminus Z)$ is an $G|_{X\setminus Z}$-torsor amounts to saying that there is an operation

$$G|_{X\setminus Z} \times_{X\setminus Z} P \to P$$

as in the previous Theorem 2.3. Since this is étale locally isomorphic to the canonical action

$$G|_{X\setminus Z} \times_{X\setminus Z} G|_{X\setminus Z} \overset{\mu}{\to} G|_{X\setminus Z}$$

which is finite étale, by faithfully flat descent the operation defines an étale covering, so extends by Zariski-Nagata purity uniquely to an étale covering $H \to X$, which by uniqueness has to be isomorphic to $G \times_X \bar{P} \to \bar{P}$. Now one has to check the condition in Theorem 2.3.

There is a finite étale Galois covering X'/X with Galois group G such that $G \times_X X'$ is isomorphic to a direct sum of μ_n with n invertible on X. The Leray spectral sequence with supports $\mathrm{H}^p(G, \mathrm{H}^q_{Z'}(X', G \times_X X')) \Rightarrow \mathrm{H}^{p+q}_Z(X,G)$ from [Kel16], p. 228, Theorem 4.9, so it suffices to show $\mathrm{H}^q_Z(X', G \times_X X') = 0$ for $q = 0,1,2$. Hence one can assume $G \cong \mu_n$ for n invertible on X.

One has an injection $\mathrm{Br}(X) \hookrightarrow \mathrm{Br}(K(X))$ with $K(X)$ the function field of X and $\mathrm{Br}(X) \to \mathrm{Br}(U) \to \mathrm{Br}(K(X))$, so $\mathrm{Br}(X) \to \mathrm{Br}(U)$ is injective. By the hypotheses on X and since the codimension of Z in X is ≥ 2, by Corollary 3.7, there is a restriction isomorphism $\mathrm{Pic}(X) \xrightarrow{\sim} \mathrm{Pic}(U)$. Hence the snake lemma applied to the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Pic}(X)/n & \longrightarrow & \mathrm{H}^2(X,\mu_n) & \longrightarrow & \mathrm{Br}(X)[n] & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Pic}(U)/n & \longrightarrow & \mathrm{H}^2(U,\mu_n) & \longrightarrow & \mathrm{Br}(U)[n] & \longrightarrow & 0
\end{array}$$

gives that $\mathrm{H}^2(X,\mu_n) \to \mathrm{H}^2(U,\mu_n)$ is injective, so $\mathrm{H}^2_Z(X,\mu_n) = 0$. □

Corollary 2.4. *Let X be a regular integral Noetherian separated scheme and $\mathscr{A}/X$ be an Abelian scheme. Let $Z \hookrightarrow X$ be a closed subscheme of codimension ≥ 2. Then $\mathrm{H}^i_Z(X,\mathscr{A})$ is torsion for all i and has trivial p-torsion for $i = 0,1,2$ and if p is invertible on X.*

Proof. By [Kel16], p. 224, Proposition 4.1, $\mathrm{H}^i(X,\mathscr{A})$ is torsion for $i > 0$. The Kummer exact sequence $0 \to \mathscr{A}[n] \to \mathscr{A} \to \mathscr{A} \to 0$ for n invertible on X yields a surjection

$$\mathrm{H}^i_Z(X,\mathscr{A}[n]) \twoheadrightarrow \mathrm{H}^i_Z(X,\mathscr{A})[n],$$

so it suffices to show that $\mathrm{H}^i_Z(X,\mathscr{A}[n]) = 0$ for $i = 1,2$. But this is Theorem 2.1. The triviality $\mathrm{H}^0_Z(X,\mathscr{A}) = 0$ is equivalent to the injectivity of

$$\mathrm{H}^0(X,\mathscr{A}) \to \mathrm{H}^0(X \setminus Z,\mathscr{A}),$$

which is clear from [Har83], p. 105, Exercise II.4.2 since $\mathscr{A}/X$ is separated, X is reduced and $X \setminus Z \hookrightarrow X$ is dense. □

With vanishing condition [Kel16, Theorem 4.5] generalizes to:

Theorem 2.5. *Let X be regular, Noetherian, integral and separated and let $\mathscr{A}$ be an Abelian scheme over X. For $x \in X$, denote the function field of X by K, the quotient field of the strict Henselization of $\mathscr{O}_{X,x}$ by K_x^{nr}, the inclusion of the generic point by $j : \{\eta\} \hookrightarrow X$ and let $j_x : \mathrm{Spec}(K_x^{nr}) \hookrightarrow \mathrm{Spec}(\mathscr{O}_{X,x}^{sh}) \hookrightarrow X$ be the composition. Then we have*

$$\mathrm{H}^1(X, \mathscr{A}) \xrightarrow{\sim} \ker\Big(\mathrm{H}^1(K, j^*\mathscr{A}) \to \prod_{x\in X} \mathrm{H}^1(K_x^{nr}, j_x^*\mathscr{A})\Big).$$

One can replace the product over all points by the following:
(a) the closed points: One has isomorphisms

$$\mathrm{H}^1(X, \mathscr{A}) \xrightarrow{\sim} \ker\Big(\mathrm{H}^1(K, j^*\mathscr{A}) \to \prod_{x\in|X|} \mathrm{H}^1(K_x^{nr}, j_x^*\mathscr{A})\Big)$$

and

$$\mathrm{H}^1(X, \mathscr{A}) \xrightarrow{\sim} \ker\Big(\mathrm{H}^1(K, j^*\mathscr{A}) \to \prod_{x\in|X|} \mathrm{H}^1(K_x^{h}, j_x^*\mathscr{A})\Big)$$

with $K_x^h = \mathrm{Quot}(\mathscr{O}_{X,x}^h)$ if $\kappa(x)$ is finite.
or (b) the codimension-1 points: One has an isomorphism

$$\mathrm{H}^1(X, \mathscr{A}) \xrightarrow{\sim} \ker\Big(\mathrm{H}^1(K, j^*\mathscr{A}) \to \bigoplus_{x\in X^{(1)}} \mathrm{H}^1(K_x^{nr}, j_x^*\mathscr{A})\Big)$$

if one disregards the p-torsion ($p = \mathrm{char}\, k$) and X/k is smooth projective over k finitely generated. For $\dim X \leq 2$, this also holds for the p-torsion.

For $x \in X^{(1)}$, one can also replace K_x^{nr} and K_x^h by the quotient field of the completions $\hat{\mathscr{O}}_{X,x}^{sh}$ and $\hat{\mathscr{O}}_{X,x}^{h}$, respectively.

3 Finiteness theorems for $\mathrm{H}^1_{\mathrm{fppf}}$ over finite fields

The aim of this section is to show that $\mathrm{H}^1_{\mathrm{fppf}}(X, G)$ is finite for X a normal proper variety over a finite field of characteristic p and G/X a finite flat group scheme.

The proof is by reduction to the case of a finite flat *simple* group scheme over an algebraically closed field, which is isomorphic to $\mathbf{Z}/\ell$ (étale-étale), $\mathbf{Z}/p$ (étale-local), μ_p (local-étale) or α_p (local-local).

We use the interpretation of $\mathrm{H}^1_{\mathrm{fppf}}(X, G)$ as G-torsors on X [Mil80, Proposition III.4.7] since G/X is affine. We also exploit de Jong's alteration theorem [de 96, Theorem 4.1].

Let us first recall some well-known facts on flat cohomology.

Definition 3.1. *An* **isogeny** *of commutative group schemes G, H of finite type over an arbitrary base scheme X is a group scheme homomorphism $f : G \to H$ such that for all $x \in X$, the induced homomorphism $f_x : G_x \to H_x$ on the fibers over x is finite and surjective on identity components.*

Remark 3.2. See [BLR90, p. 180, Definition 4]. We will usually consider isogenies between abelian schemes, for example the finite flat n-multiplication, which is étale iff n is invertible on the base scheme or the abelian schemes are trivial.

Lemma 3.3. *Let G, G' be commutative group schemes over a scheme X which are smooth and of finite type over X with connected fibers and $\dim G = \dim G'$ and let $f : G' \to G$ be a morphism of commutative group schemes over X.*

If f is flat (respectively, étale) then $\ker(f)$ is a flat (respectively, étale) group scheme over X, f is quasi-finite, surjective and defines an epimorphism in the category of flat (respectively, étale) sheaves over X.

Proof. See [Kel19, Lemma 2.3.3]. □

Lemma 3.4 (Kummer sequence)**.** *Let $f : G \to G'$ be a faithfully flat isogeny between smooth commutative group schemes over a base scheme X. Then the sequence*

$$0 \to \ker(f) \to G \xrightarrow{f} G' \to 0$$

is exact on X_{fppf}. This applies in particular to $G = \mathbf{G}_m$ and $G = \mathscr{A}$ an abelian scheme and the n-multiplication morphism for arbitrary $n \neq 0$.

Proof. Since f is faithfully flat, in particular surjective, it is an epimorphism of sheaves by Lemma 3.3. An isogeny of abelian schemes is faithfully flat by [Mil86, Proposition 8.1]. □

Lemma 3.5. *Let X/S be a* smooth *commutative group scheme. Then there are comparison isomorphisms*

$$\mathrm{H}^i_{\mathrm{fppf}}(X, G) = \mathrm{H}^i_{\text{ét}}(X, G).$$

In particular, $\mathrm{H}^i_{\mathrm{fppf}}(X, G)$ is finite if X is proper over a finite field and G is a commutative finite étale group scheme.

Proof. See [Mil80, Remark III.3.11 (b)] and note that the proof given there gives a comparison isomorphism for any topologies between the étale and the flat site. □

Lemma 3.6. *Let X be a Noetherian integral scheme with function field $K(X)$ and $U \subseteq X$ dense open. Then there is an exact sequence*

$$1 \to \mathbf{G}_m(X) \to \mathbf{G}_m(U) \to \bigoplus_{D \in (X \setminus U)^{(1)}} \mathbf{Z}[D] \to \mathrm{Cl}(X) \to \mathrm{Cl}(U) \to 0.$$

Proof. The assumptions imply that there is a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathscr{O}_X(X)^\times & \longrightarrow & K(X)^\times & \longrightarrow & \mathrm{Div}(X) & \longrightarrow & \mathrm{Cl}(X) & \longrightarrow & 0 \\
 & & \downarrow & & \| & & \twoheaddownarrow & & \downarrow & & \\
1 & \longrightarrow & \mathscr{O}_X(U)^\times & \longrightarrow & K(U)^\times & \longrightarrow & \mathrm{Div}(U) & \longrightarrow & \mathrm{Cl}(U) & \longrightarrow & 0.
\end{array}$$

A diagram chase yields the result. □

Corollary 3.7. *Let X be a Noetherian integral regular scheme and let $U \subseteq X$ be dense open. Then there is an exact sequence*

$$1 \to \mathbf{G}_m(X) \to \mathbf{G}_m(U) \to \bigoplus_{D \in (X \setminus U)^{(1)}} \mathbf{Z}[D] \to \mathrm{Pic}(X) \to \mathrm{Pic}(U) \to 0.$$

Proof. By the assumptions, $\mathrm{Cl}(X) = \mathrm{Pic}(X)$ and $\mathrm{Cl}(U) = \mathrm{Pic}(U)$. □

Corollary 3.8. *Let $X/\mathbf{F}_q$ be an integral Noetherian regular proper variety and let $j : U \hookrightarrow X$ be the inclusion of an open subscheme of X. Then $\mathrm{H}^1_{\mathrm{fppf}}(U, \mu_{p^n})$ is finite for all n and any prime p.*

Proof. The Kummer sequence Lemma 3.4 on U_{fppf} together with $\mathrm{Pic}(U) = \mathrm{H}^1_{\mathrm{fppf}}(U, \mathbf{G}_{m,U})$ by Lemma 3.5 yields the exact sequence

$$1 \to \mathbf{G}_m(U)/p^n \to \mathrm{H}^1_{\mathrm{fppf}}(U, \mu_{p^n}) \to \mathrm{Pic}(U)[p^n] \to 0.$$

Since $\mathbf{G}_m(X) = \Gamma(X, \mathbf{G}_m)^\times$ is finite by the coherence theorem since $X/\mathbf{F}_q$ is proper and $\mathbf{F}_q$ is finite, and since $\mathrm{Pic}(X)$ is finitely generated since its sits in a short exact sequence $0 \to \mathrm{Pic}^0(X) \to \mathrm{Pic}(X) \to \mathrm{NS}(X) \to 0$ and $\mathrm{Pic}^0(X)$ is finite since it is the group of rational points of an Abelian variety over a finite field and $\mathrm{NS}(X)$ is always finitely generated by [Mil80], p. 215, Theorem V.3.25, by Corollary 3.7 and the finiteness of $(X \setminus U)^{(1)}$, this exact sequence gives the finiteness of $\mathbf{G}_m(U)/p^n$ and of $\mathrm{Pic}(U)[p^n]$. □

Lemma 3.9. *Let X be a normal integral scheme and G/X be a finite flat group scheme. If T is a G-torsor on X trivial over the generic point of X, then T is trivial. Hence, $\mathrm{H}^1_{\mathrm{fppf}}(X, G) \to \mathrm{H}^1_{\mathrm{fppf}}(K(X), G)$ is injective, and if $f : Y \to X$ is birational, $f^* : \mathrm{H}^1_{\mathrm{fppf}}(X, G) \to \mathrm{H}^1_{\mathrm{fppf}}(Y, G)$ is injective.*

Proof. Since T is trivial over the generic point of X, generically, there is a section of $\pi : T \to X$. This extends to a rational map $\sigma : X \dashrightarrow T$. Take the schematic closure $i : X' \hookrightarrow T$ of σ. The composition $\pi \circ i : X' \to T \to X$ is birational and finite (as a composition of a closed immersion and a finite morphism). By [GW10, Corollary 12.88], since X is normal, $X' \to X$ is an isomorphism. Hence σ is a section of π, so T/X is trivial. □

Lemma 3.10. *Let X be a proper variety over a finite field and Y/X be a finite flat scheme. Let Z/X be proper. Then $Y(Z)$ is finite.*

Proof. Since $\mathrm{Mor}_X(Z, Y) = \mathrm{Mor}_Z(Z, Y \times_X Z)$, one can assume $Z = X$. So we have to show that there are only finitely many sections to $\pi : Y \to X$. Such a section corresponds to an $\mathscr{O}_X$-algebra map $\pi_*\mathscr{O}_Y \to \mathscr{O}_X$. But $\mathrm{H}^0_{\mathrm{Zar}}(X, \mathscr{H}\mathrm{om}_X(\pi_*\mathscr{O}_Y, \mathscr{O}_X))$ is finite by the coherence theorem as it is a finite dimensional vector space over a finite field. □

Lemma 3.11. *Let $Y \to X$ be an alteration of proper integral varieties with X normal, and G/X be a finite flat commutative group scheme. Then $\ker(\mathrm{H}^1_{\mathrm{fppf}}(X, G) \to \mathrm{H}^1_{\mathrm{fppf}}(Y, G))$ is finite. Hence $\mathrm{H}^1_{\mathrm{fppf}}(X, G)$ is finite if $\mathrm{H}^1_{\mathrm{fppf}}(Y, G)$ is.*

Proof. If $Y \to X$ is a blow-up, the kernel is trivial by Lemma 3.9 since a blow-up is birational. Hence the statement holds for blow-ups.

Since a normalization morphism of integral schemes is birational [Liu06, Proposition 4.1.22], one can assume X' normal.

By [RG71, Théorème 5.2.2], there is a blow-up $X' \to X$ such that $Y' := Y \times_X X'$ is flat over X'. There is a commutative diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & \ker(\mathrm{H}^1_{\mathrm{fppf}}(X, G) \to \mathrm{H}^1_{\mathrm{fppf}}(Y, G)) & \longrightarrow & \mathrm{H}^1_{\mathrm{fppf}}(X, G) \\ & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \ker(\mathrm{H}^1_{\mathrm{fppf}}(X', G) \to \mathrm{H}^1_{\mathrm{fppf}}(Y', G)) & \longrightarrow & \mathrm{H}^1_{\mathrm{fppf}}(X', G) \end{array}$$

By the snake lemma, since $\ker(\mathrm{H}^1_{\mathrm{fppf}}(X, G) \to \mathrm{H}^1_{\mathrm{fppf}}(X', G))$ is finite as $X' \to X$ is a blow-up, $\ker(\mathrm{H}^1_{\mathrm{fppf}}(X, G) \to \mathrm{H}^1_{\mathrm{fppf}}(Y, G))$ is finite if we can show that $\ker(\mathrm{H}^1_{\mathrm{fppf}}(X', G) \to \mathrm{H}^1_{\mathrm{fppf}}(Y', G))$ is finite. Hence, we can assume $Y \to X$ finite flat.

Let $T \to X$ be in the kernel, i. e. it is a torsor trivial on Y. Choose a section $\sigma : Y \to T \times_X Y$. Since $T \times_X (Y \times_X Y) \to Y \times_X Y$ is a G-torsor, one can take

$$\tau := \sigma \circ \mathrm{pr}_0 - \sigma \circ \mathrm{pr}_1 \in G(Y \times_X Y).$$

The section τ corresponds to the isomorphism class of the G-torsor T by descent theory for the fppf covering $\{Y \to X\}$, but by Lemma 3.10, $G(Y \times_X Y)$ is finite. □

Lemma 3.12. *Let X be an integral scheme with function field K and G/X be a finite flat group scheme. Let $H_K \hookrightarrow G_K$ be a finite flat group scheme. Then there is a blow-up $\tilde{X}/X$ such that H_K extends to a finite flat subgroup scheme of $G \times_X \tilde{X}$.*

Proof. Let $H \hookrightarrow G$ be the schematic closure of $H_K \hookrightarrow G$. The morphism $H \to G \to X$ is finite as a composition of a closed immersion and a finite morphism. By [RG71, Théorème 5.2.2], there is a blow-up $X' \to X$ such that $H' := H \times_X X' \to X'$ is flat. Then, H' is the schematic closure of $H_K \hookrightarrow G' := G \times_X X'$. So one can assume H/X finite flat.

Let $Y \to X$ be finite flat. Since the morphism is affine, locally, one has the diagram

$$\begin{array}{ccc} A & \hookrightarrow & A \otimes_R \mathrm{Quot}(R) \\ \uparrow & & \uparrow \\ R & \hookrightarrow & \mathrm{Quot}(R). \end{array}$$

Here, the upper horizontal arrow is injective by flatness of $R \to A$. Hence Y is the schematic closure of Y_K in Y.

By flatness, the schematic closure of $H_K \times_K H_K$ in $G \times_X G$ is $H \times_X H$. By the universal property of the schematic closure [GW10, (10.8)], one has the factorization

$$\begin{array}{ccc} H_K \times_K H_K^{\mu} & \longrightarrow & H_K \\ \downarrow & & \downarrow \\ H \times_X H^{\mu} & \dashrightarrow & H \\ \downarrow & & \downarrow \\ G \times_X G^{\mu} & \longrightarrow & G, \end{array}$$

for the multiplication μ, and similar for the inverse and unit section. □

Lemma 3.13. *Let X be a proper integral variety over a field and G/X be a finite flat commutative group scheme. After an alteration $X' \to X$, there exists a filtration of G by finite flat group schemes with subquotients of prime order.*

Proof. Over the algebraic closure of the function field of X, there is such an filtration since the only simple objects in the category of finite flat group schemes of p-power order are μ_p, $\mathbf{Z}/p$ and α_p. Since everything is of finite presentation, these are defined over a finite extension of the function field [GW10, Corollary 10.79]. Now take the normalization in this finite extension of function fields and use Lemma 3.12. □

Theorem 3.14. *Let X be a proper integral normal variety over a finite field and G/X be a finite flat commutative group scheme. Then $\mathrm{H}^1_{\mathrm{fppf}}(X, G)$ is finite.*

Proof. By Lemma 3.13, Lemma 3.11 and the long exact cohomology sequence one can assume G of prime order p (since the case of G/X étale is easily dealt with). Since then G is simple by [Sha86, p. 38] and since $F \circ V = [p] = 0$ by [Sha86, p. 62] and [Mum70, p. 141], either $V = 0$ or $F = 0$ on G.

If $V = 0$, by [de 93, Proposition 2.2], there is a short exact sequence

$$0 \to G \to \mathscr{L} \to \mathscr{M} \to 0$$

with vector bundles $\mathscr{L}, \mathscr{M}$. By the coherence theorem, as X is proper and lives over a finite ground field, and by comparison of Zariski and fppf cohomology [Mil80, Proposition III.3.7], the long exact cohomology sequence shows that $\mathrm{H}^i_{\mathrm{fppf}}(X, G)$ is finite.

If $F = 0$, after replacing X by an alteration by Lemma 3.11 as in the proof of Lemma 3.13, one can assume that G is isomorphic to μ_p over the generic point. Since for $Y, Z/X$ of finite presentation such that $Y_K \cong Z_K$, there is a non-empty open subscheme $U \hookrightarrow X$ such that $Y_U \cong Z_U$, there is a non-empty open subscheme $U \hookrightarrow X$ such that $G_U \cong \mu_{p,U}$. There is an alteration $f : X' \to X$ such that X' is regular. By Corollary 3.8, $\mathrm{H}^1_{\mathrm{fppf}}(f^{-1}(U), \mu_p)$ is finite. By Lemma 3.9, $\mathrm{H}^1_{\mathrm{fppf}}(X', G \times_X X')$ is finite, so by Lemma 3.11, $\mathrm{H}^1_{\mathrm{fppf}}(X, G)$ is finite. □

4 Isogeny invariance of finiteness of Ш, the p-part

In this section, we extend [Kel16], p. 240, Theorem 4.31 to p^∞-torsion.

Theorem 4.1. *Let X/k be a proper variety over a finite field k and $f : \mathscr{A} \to \mathscr{A}'$ be an isogeny of Abelian schemes over X. Let p be an arbitrary prime. Assume f étale if $p \neq \operatorname{char} k$. Then $Ш(\mathscr{A}/X)[p^\infty]$ is finite if and only if $Ш(\mathscr{A}'/X)[p^\infty]$ is finite.*

Proof. In the case where ℓ is invertible on X and f is étale (i. e., of degree invertible on X), this is [Kel16], p. 240, Theorem 4.31.

Now assume $p = \operatorname{char} k$. The short exact sequence of flat sheaves Lemma 3.4 yields an exact sequence in cohomology

$$\mathrm{H}^1_{\mathrm{fppf}}(X, \ker(f)) \to \mathrm{H}^1_{\mathrm{fppf}}(X, \mathscr{A}) \xrightarrow{f} \mathrm{H}^1_{\mathrm{fppf}}(X, \mathscr{A}')$$

and note that $\mathrm{H}^1_{\mathrm{fppf}}(X, \mathscr{A}) = \mathrm{H}^1_{\text{ét}}(X, \mathscr{A}) = Ш(\mathscr{A}/X)$ by Lemma 3.5 since $\mathscr{A}/X$ is smooth, and that $\mathrm{H}^1_{\mathrm{fppf}}(X, \ker(f))$ is finite by Theorem 3.14. Note that all groups are torsion (the Tate-Shafarevich groups by [Kel16], p. 224, Proposition 4.1), hence the sequence stays exact after taking p^∞-torsion. So $Ш(\mathscr{A}/X)[p^\infty]$ is finite if $Ш(\mathscr{A}'/X)[p^\infty]$ is.

For the converse, note that by [Kel19], Proposition 2.19 there is a polarization $\lambda : \mathscr{A}^t \to \mathscr{A}$. Hence, the argument above for λ and λ^t implies that $Ш(\mathscr{A}^t/X)[p^\infty]$ is finite iff $Ш(\mathscr{A}/X)[p^\infty]$ is, and analogously for $Ш(\mathscr{A}'/X)[p^\infty]$. Taking the dual Kummer sequence $0 \to \ker(f^t) \to \mathscr{A}'^t \to \mathscr{A}^t \to 0$ yields an exact sequence

$$\mathrm{H}^1_{\mathrm{fppf}}(X, \ker(f^t)) \to Ш(\mathscr{A}'^t/X) \to Ш(\mathscr{A}^t/X).$$

By the same argument as above, $Ш(\mathscr{A}'^t/X)[p^\infty]$ is finite if $Ш(\mathscr{A}^t/X)[p^\infty]$ is if $Ш(\mathscr{A}/X)[p^\infty]$ is. So $Ш(\mathscr{A}'/X)[p^\infty]$ is finite. ☐

5 Descent of finiteness of Ш, the p-part

In this section, we extend [Kel16], p. 238, Theorem 4.29 to p^∞-torsion.

Lemma 5.1. *Let $\mathscr{A}/X$ be an Abelian scheme over a proper variety X over a finite field of characteristic p. Then $Ш(\mathscr{A}/X)[p^\infty]$ is cofinitely generated.*

Recall that $Ш(\mathscr{A}/X)$ was defined as $\mathrm{H}^1_{\text{ét}}(X, \mathscr{A})$ in [Kel16], p. 225, Definition 4.2.

Proof. The long exact cohomology sequence associated to the Kummer sequence Lemma 3.4 gives us a surjection

$$\mathrm{H}^1_{\mathrm{fppf}}(X, \mathscr{A}[p^n]) \twoheadrightarrow \mathrm{H}^1_{\mathrm{fppf}}(X, \mathscr{A})[p^n] \to 0$$

Now, since $\mathscr{A}/X$ is a smooth group scheme, Lemma 3.5 gives us an isomorphism $\mathrm{H}^1_{\mathrm{fppf}}(X, \mathscr{A}) = \mathrm{H}^1_{\text{ét}}(X, \mathscr{A})$, which by definition equals $Ш(\mathscr{A}/X)$. By Theorem 3.14, $\mathrm{H}^1_{\mathrm{fppf}}(X, \mathscr{A}[p^n])$ is finite since $X/\mathbf{F}_q$ is proper. From this, one sees that $\mathrm{H}^1_{\text{ét}}(X, \mathscr{A})[p]$ is finite. Hence $Ш(\mathscr{A}/X)[p^\infty]$ is cofinitely generated by [Kel19], Lemma 2.38. ☐

Lemma 5.2. *Let $f : X' \to X$ be a finite étale morphism of constant degree d and let $\mathscr{F}$ be an fppf sheaf on X. Then there is a trace map $\mathrm{Tr}_f : f_*f^*\mathscr{F} \to \mathscr{F}$, functorial in $\mathscr{F}$, such that $\varphi \mapsto \mathrm{Tr}_f \circ f_*(\varphi)$ is an isomorphism $\mathrm{Hom}_{X'}(\mathscr{F}', f^*\mathscr{F}) \to \mathrm{Hom}_X(\pi_*\mathscr{F}', \mathscr{F})$ for any fppf sheaf $\mathscr{F}'$ on X'. Thus, $f_* = f_!$, that is, f_* is left adjoint to f^*, and Tr_f is the adjunction map. The composites*

$$\mathscr{F} \to f_*f^*\mathscr{F} \overset{\mathrm{Tr}_f}{\to} \mathscr{F} \quad \text{and} \quad \mathrm{H}^r_{\mathrm{fppf}}(X, \mathscr{F}) \overset{f^*}{\to} \mathrm{H}^r_{\mathrm{fppf}}(X', f^*\mathscr{F}) \overset{\mathrm{can}}{\to} \mathrm{H}^r_{\mathrm{fppf}}(X, f_*f^*\mathscr{F}) \overset{\mathrm{Tr}_f}{\to} \mathrm{H}^r_{\mathrm{fppf}}(X, \mathscr{F})$$

are multiplication by d.

Proof. As in [Mil80], p. 168, Lemma V.1.12. ☐

Theorem 5.3. *Let p be a prime and X be a scheme of characteristic p. Let $f : X' \to X$ be a proper, surjective, generically étale morphism of generical degree prime to p of integral, normal varieties over a finite field. Let $\mathscr{A}$ be an abelian scheme on X and $\mathscr{A}' := f^*\mathscr{A} = \mathscr{A} \times_X X'$. If $Ш(\mathscr{A}'/X')[p^\infty]$ is finite, so is $Ш(\mathscr{A}/X)[p^\infty]$.*

Proof. The same proof as in [Kel16, Theorem 4.29] works, one only needs $Ш(\mathscr{A}/X)[p^\infty]$ to be cofinitely generated in Step 2, which is Lemma 5.1. The trace morphism in Step 3 for fppf cohomology comes from Lemma 5.2. Note that the proof given there does not need the regularity of X, X' and that varieties over a field are excellent by [Liu06, Corollary 2.40 (a)]. ☐

Lemma 5.4. *Let ℓ be a prime number and A, B be cofinitely generated abelian ℓ-primary torsion groups. Assume that there are morphisms $f : A \to B$ and $g : B \to A$ such that $g \circ f : A \to A$ is multiplication by a non-zero integer n. Then the $\ell\text{-}\mathrm{rk}\, A \leq \ell\text{-}\mathrm{rk}\, B$.*

Proof. Take the image of our situation modulo the Serre subcategory (!) of finite ℓ-torsion groups. Then $g \circ f = [n]$ is an isomorphism in the quotient category (it has trivial kernel and cokernel). Hence f is a split monomorphism, so A is a direct summand in B. Because the ℓ-rank is additive, the claim follows. ☐

Corollary 5.5. *In the situation of the lemma above, if B is finite, A is finite.*

Proof. A cofinitely generated abelian ℓ-primary torsion group is finite iff its ℓ-rank is 0. ☐

We remove the hypotheses that f is generically étale and has degree prime to ℓ in [Kel16, Theorem 4.29]:

Theorem 5.6. *Let $f : X' \to X$ be a proper, surjective, generically finite morphism of integral, normal varieties over a finite field. Let $\mathscr{A}$ be an abelian scheme on X and $\mathscr{A}' := f^*\mathscr{A} = \mathscr{A} \times_X X'$. Let ℓ be invertible on X. If $Ш(\mathscr{A}'/X')[\ell^\infty]$ is finite, so is $Ш(\mathscr{A}/X)[\ell^\infty]$.*

Proof. By the Stein factorization, f factors as a proper, surjective, birational morphism followed by a finite morphism. In particular, it is a generically étale alteration. The finite morphism factors as a finite purely inseparable morphism followed by a finite generically étale morphism.

If f is generically étale: Upgrade the proof of Step 2 of [Kel16, Theorem 4.29] with the previous lemma to remove the assumption on the generical degree of f.

If f is a universal homeomorphism, $f^* : \mathrm{H}^1_{\text{ét}}(X, \mathscr{A}[\ell^n]) \to \mathrm{H}^1_{\text{ét}}(X', f^*\mathscr{A}[\ell^n])$ is an isomorphism by [SGA4.2, VIII.1.1]. But $\mathscr{A}[\ell^n]/X$ is étale because ℓ is invertible on X, so $\mathrm{H}^1_{\text{ét}}(X', \mathscr{A}'[\ell^n])$. □

6 The p-part of the Brauer group

Let X be a smooth projective geometrically integral variety over a finite field $k = \mathbf{F}_q$ of characteristic p with absolute Galois group Γ topologically generated by the Frobenius Frob. By [Kel16], p. 213, Corollary 2.5, one has $\mathrm{Br}(X) = \mathrm{H}^2_{\text{ét}}(X, \mathbf{G}_m) = \mathrm{H}^2_{\text{ét}}(X, \mathbf{G}_m)_{\text{tors}}$, and these equal the corresponding flat cohomology groups by Lemma 3.5.

Proposition 6.1. *There is a diagram of finitely generated $\mathbf{Z}_p$-modules with exact rows and columns*

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \mathrm{Pic}(X) \otimes_{\mathbf{Z}} \mathbf{Z}_p & \longrightarrow & (\mathrm{NS}(\bar{X}) \otimes_{\mathbf{Z}} \mathbf{Z}_p)^\Gamma & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{H}^1_{\text{fppf}}(\bar{X}, \mathbf{Z}_p(1))_\Gamma & \longrightarrow & \mathrm{H}^2_{\text{fppf}}(X, \mathbf{Z}_p(1)) & \longrightarrow & \mathrm{H}^2_{\text{fppf}}(\bar{X}, \mathbf{Z}_p(1))^\Gamma & \longrightarrow & 0 \\
 & & & & \downarrow & & & & \\
 & & & & T_p \,\mathrm{Br}(X) & & & & \\
 & & & & \downarrow & & & & \\
 & & & & 0. & & & &
\end{array}$$

The group $\mathrm{H}^1_{\text{fppf}}(\bar{X}, \mathbf{Z}_p(1))_\Gamma$ is killed by multiplication by $t(X) := |\,\mathrm{Pic}(X)_{\text{tors}}| = |\,\mathrm{Pic}(\bar{X})^\Gamma_{\text{tors}}| < \infty$.

Proof. Pass to the projective limit in

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \mathrm{Pic}(X)/p^n & \longrightarrow & (\mathrm{NS}(\bar{X})/p^n)^\Gamma & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{H}^1_{\text{fppf}}(\bar{X}, \mu_{p^n})_\Gamma & \longrightarrow & \mathrm{H}^2_{\text{fppf}}(X, \mu_{p^n}) & \longrightarrow & \mathrm{H}^2_{\text{fppf}}(\bar{X}, \mu_{p^n})^\Gamma & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathrm{Pic}(\bar{X})[p^n]_\Gamma & & \mathrm{Br}(X)[p^n] & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0. & & & &
\end{array}$$

The columns come from the Kummer sequence Lemma 3.4, and the middle row comes from the Hochschild-Serre spectral sequence for the flat topology [Mil80], p. 105, Remark III.2.21 (a). Note that $\mathrm{H}^2_{\text{fppf}}(X, \mathbf{Z}_p(1))$ is a finitely generated $\mathbf{Z}_p$-module by [Ill79], p. 629, Proposition 5.9.

One has $\mathrm{Pic}(X) = \mathrm{Pic}(\bar{X})^\Gamma$ since the Hochschild-Serre spectral sequence for the flat topology [Mil80], p. 105, Remark III.2.21 (a) gives us an exact sequence

$$0 \to \mathrm{H}^1(\Gamma, \mathrm{H}^0_{\mathrm{fppf}}(\bar{X}, \mathbf{G}_m)) \to \mathrm{Pic}(X) \to \mathrm{Pic}(\bar{X})^\Gamma \to \mathrm{H}^2(\Gamma, \mathrm{H}^0_{\mathrm{fppf}}(\bar{X}, \mathbf{G}_m)),$$

and $\mathrm{H}^0_{\mathrm{fppf}}(\bar{X}, \mathbf{G}_m) = \bar{k}^\times$, and $\mathrm{H}^1(\Gamma, \bar{k}^\times) = 0$ by Hilbert's theorem 90 and $\mathrm{H}^2(\Gamma, \bar{k}^\times) = \mathrm{Br}(k) = 0$. The order of the group $(\mathrm{Pic}(\bar{X})[p^n])_\Gamma$ equals the order of the group $(\mathrm{Pic}(\bar{X})[p^n])^\Gamma$ since $\mathrm{Pic}(\bar{X})[p^n]$ is finite, so its Herbrand quotient equals 1. Hence $\mathrm{H}^1_{\mathrm{fppf}}(\bar{X}, \mu_{p^n})_\Gamma$ is killed by multiplication with $t(X)$, so $\mathrm{H}^1_{\mathrm{fppf}}(\bar{X}, \mathbf{Z}_p(1))_\Gamma$ is killed by multiplication by $t(X)$. □

Proposition 6.2. *There is a diagram*

$$\begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ & & \mathrm{Pic}(X) \otimes_{\mathbf{Z}} \mathbf{Q}_p & \longrightarrow & (\mathrm{NS}(\bar{X}) \otimes_{\mathbf{Z}} \mathbf{Q}_p)^\Gamma & & \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{H}^2_{\mathrm{fppf}}(X, \mathbf{Q}_p(1)) & \longrightarrow & \mathrm{H}^2_{\mathrm{fppf}}(\bar{X}, \mathbf{Q}_p(1))^\Gamma & \longrightarrow & 0 \\ & & \downarrow & & & & \\ & & V_p \, \mathrm{Br}(X) & & & & \\ & & \downarrow & & & & \\ & & 0 & & & & \end{array}$$

of finitely generated $\mathbf{Q}_p$-vector spaces with exact rows and columns.

Proof. Tensor the groups in Proposition 6.1 with $\mathbf{Q}_p$ over $\mathbf{Z}_p$. Since $\mathrm{H}^1_{\mathrm{fppf}}(\bar{X}, \mathbf{Z}_p(1))_\Gamma$ is killed by multiplication with $t(X)$, it is 0 after tensoring with $\mathbf{Q}_p$. The upper horizontal arrow is an isomorphism since $\mathrm{Pic}(X) = \mathrm{Pic}(\bar{X})^\Gamma$ (see the proof of Proposition 6.1) and $\mathrm{Pic}(\bar{X}) \otimes_{\mathbf{Z}} \mathbf{Q}_p = \mathrm{NS}(\bar{X}) \otimes_{\mathbf{Z}} \mathbf{Q}_p$ because $\mathrm{Pic}^0(\bar{X}) = \mathbf{Pic}^0_{X/k}(\bar{k}) = \varinjlim_n \mathbf{Pic}^0_{X/k}(\mathbf{F}_{q^n})$ is torsion as a colimit of finite groups; furthermore, $(\mathrm{NS}(\bar{X}) \otimes_{\mathbf{Z}} \mathbf{Q}_p)^\Gamma = \mathrm{NS}(\bar{X})^\Gamma \otimes_{\mathbf{Z}} \mathbf{Q}_p$ since $- \otimes_{\mathbf{Z}} \mathbf{Q}_p$ is exact and $\mathrm{NS}(\bar{X})$ is a discrete Γ-module and for a finite group G, one has $A^G = \varprojlim(A \to \oplus_{g \in G} A)$. □

Corollary 6.3. *The following are equivalent:*

1. *The group* $\mathrm{Br}(X)[p^\infty]$ *is finite.*
2. $\mathrm{Br}(X)[p^\infty]_{\mathrm{div}} = 0$ *and* $\mathrm{Br}(X)[p^\infty] = \mathrm{Br}(X)[p^\infty]_{\text{n-div}}$.
3. $V_p \, \mathrm{Br}(X) = 0$.
4. $\mathrm{NS}(X) \otimes_{\mathbf{Z}} \mathbf{Q}_p = \mathrm{H}^2_{\mathrm{fppf}}(\bar{X}, \mathbf{Q}_p(1))^\Gamma = \mathrm{H}^2_{\mathrm{fppf}}(\bar{X}, \mathbf{Q}_p(1))^{\mathrm{Frob}}$.
5. $\dim_{\mathbf{Q}_p} \mathrm{H}^2_{\mathrm{fppf}}(\bar{X}, \mathbf{Q}_p(1))^\Gamma = \rho(X)$ *with* $\rho(X) := \mathrm{rk}_{\mathbf{Z}} \mathrm{NS}(X) = \mathrm{rk}_{\mathbf{Z}} \mathrm{Pic}(X)$.

One always has $\dim_{\mathbf{Q}_p} \mathrm{H}^2_{\mathrm{fppf}}(\bar{X}, \mathbf{Q}_p(1))^\Gamma \geq \rho(X)$.

Lemma 6.4. *Let k be a finite field of characteristic p and A/k be an Abelian variety. Then the action of $\Gamma = \mathrm{Gal}(\bar{k}/k)$ on $T_p A$ is semisimple.*

Proof. Let $\ell \neq p$ be prime. The minimal polynomial of the Frobenius acting on $T_\ell A$ is defined over $\mathbf{Q}$ since it equals the radical of the characteristic polynomial, which is defined over $\mathbf{Q}$ by [Mil86], p. 125, Proposition 12.9.

By [Tat66], the Frobenius acts semisimply on $T_\ell A$ for $\ell \neq p$ prime, and hence its minimal polynomial is square-free. Since $\mathrm{End}_k(A) \otimes_{\mathbf{Z}} \mathbf{Q}_\ell \to \mathrm{End}_{\mathbf{Q}_\ell}(T_\ell A)$ is injective, it also satisfies this square-free polynomial in $\mathrm{End}_k(A) \otimes_{\mathbf{Z}} \mathbf{Q} \subset \mathrm{End}_k(A) \otimes_{\mathbf{Z}} \mathbf{Q}_\ell$, so it is semisimple. Hence it also acts semisimply on $T_p A$ as there is an injection $\mathrm{End}_k(A) \otimes_{\mathbf{Z}} \mathbf{Q}_p \hookrightarrow \mathrm{End}_{\mathbf{Q}_p}(T_p A)$. □

Theorem 6.5. *Let X be a product of smooth proper curves and Abelian varieties over a finite field k of characteristic p. Then $\dim_{\mathbf{Q}_p} \mathrm{H}^2_{\mathrm{cris}}(\bar{X}/W)^\Gamma \otimes \mathbf{Q}_p = \dim_{\mathbf{Q}_p} \mathrm{H}^2_{\mathrm{fppf}}(X, \mathbf{Q}_p(1)) = \rho(X)$. Hence $\mathrm{Br}(X)[p^\infty]$ is finite by Corollary 6.3.*

Proof. Note that proper curves and Abelian varieties are projective by [Har83], p. 136, Proposition II.6.7 and [Mil86], p. 113, Theorem 7.1. By Corollary 6.3,

$$\dim_{\mathbf{Q}_p} \mathrm{H}^2_{\mathrm{fppf}}(X, \mathbf{Q}_p(1)) \geq \rho(X),$$

so it suffices to prove that $\dim_{\mathbf{Q}_p} \mathrm{H}^2_{\mathrm{fppf}}(X, \mathbf{Q}_p(1)) \leq \rho(X)$.

One has for $\ell \neq p$ prime

$$\rho(X) = \dim_{\mathbf{Q}_\ell} \mathrm{H}^2_{\text{ét}}(\bar{X}, \mathbf{Q}_\ell(1))^\Gamma = \dim_{\mathbf{Q}_p} \mathrm{H}^2_{\mathrm{cris}}(X/W) \otimes_{\mathbf{Z}} \mathbf{Q}_p,$$

the first equality by the Tate conjecture [Tat66, Theorem 4], and the second equality by [KM74, Corollary 1 1)] since X/k is smooth and projective.

One has $\mathrm{H}^{2r}_{\text{ét}}(\bar{X}, \mathbf{Q}_\ell(i))^\Gamma = \mathrm{H}^{2r}_{\text{ét}}(X, \mathbf{Q}_\ell(i))$ for k finite: The Hochschild-Serre spectral sequence

$$\mathrm{H}^p(\Gamma, \mathrm{H}^q_{\text{ét}}(\bar{X}, \mathbf{Q}_\ell(i))) \Longrightarrow \mathrm{H}^{p+q}_{\text{ét}}(X, \mathbf{Q}_\ell(i))$$

yields by $\mathrm{cd}(k) = 1$ a short exact sequence

$$0 \to \mathrm{H}^1(\Gamma, \mathrm{H}^{q-1}_{\text{ét}}(\bar{X}, \mathbf{Q}_\ell(i))) \to \mathrm{H}^q_{\text{ét}}(X, \mathbf{Q}_\ell(i)) \to \mathrm{H}^q_{\text{ét}}(\bar{X}, \mathbf{Q}_\ell(i))^\Gamma \to 0.$$

But $\mathrm{H}^{q-1}_{\text{ét}}(\bar{X}, \mathbf{Q}_\ell(i))$ is uniquely divisible, so $\mathrm{H}^1(\Gamma, \mathrm{H}^{q-1}_{\text{ét}}(\bar{X}, \mathbf{Q}_\ell(i))) = 0$. Analogously, one has $\mathrm{H}^q_{\mathrm{fppf}}(X, \mathbf{Q}_p(i)) = \mathrm{H}^q_{\mathrm{fppf}}(\bar{X}, \mathbf{Q}_p(i))^\Gamma$.

Furthermore, $\dim_{\mathbf{Q}_\ell} \mathrm{H}^2_{\text{ét}}(\bar{X}, \mathbf{Q}_\ell(1)) = \dim_{\mathbf{Q}_p} \mathrm{H}^2_{\mathrm{cris}}(\bar{X}/W) \otimes_{\mathbf{Z}_p} \mathbf{Q}_p \geq \dim_{\mathbf{Q}_p} \mathrm{H}^2_{\mathrm{fppf}}(\bar{X}, \mathbf{Q}_p(1))$, so

$$\rho(X) = \dim_{\mathbf{Q}_\ell} \mathrm{H}^2_{\text{ét}}(\bar{X}, \mathbf{Q}_\ell(1))^\Gamma \geq \dim_{\mathbf{Q}_p} \mathrm{H}^2_{\mathrm{fppf}}(X, \mathbf{Q}_p(1)).$$

On the other hand,

$$\dim_{\mathbf{Q}_p} \mathrm{H}^2_{\mathrm{fppf}}(X, \mathbf{Q}_p(1)) \leq \dim_{\mathbf{Q}_p} \mathrm{H}^2_{\mathrm{cris}}(X/W) \otimes_{\mathbf{Z}} \mathbf{Q}_p$$

by [Ill79, Théorème 5.5 (5.5.3) or Théorème 5.14].

By Tate's conjecture [Tat66] the Frobenius action on the étale cohomology $\mathrm{H}^2_{\text{ét}}(\bar{X}, \mathbf{Q}_\ell(1)) = \bigwedge^2 \mathrm{H}^1_{\text{ét}}(\bar{X}, \mathbf{Q}_\ell) \otimes \mathbf{Q}_\ell(1)$ is semisimple, hence $\dim_{\mathbf{Q}_\ell} \mathrm{H}^2_{\text{ét}}(\bar{X}, \mathbf{Q}_\ell(1))^\Gamma$ is equal to the multiplicity of $1 - X$ as a factor of the characteristic polynomial of the Frobenius morphism, which is independent of ℓ. By Lemma 6.4 this also holds for crystalline cohomology (which is a Weil cohomology theory). □

Corollary 6.6. *The Brauer group of a product of smooth proper curves and abelian varieties over a finite field is finite.*

Proof. Combine Theorem 6.5 with [Zar83], p. 214, Corollary 2.3.5 and use that the Brauer group of a regular scheme is torsion. □

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